3-1 POWER FUNCTIONS

Before we start to analyze polynomials of degree higher than two (quadratics), we first will look at very simple functions known as **power functions**. The formal definition of a power function is given below:

POWER FUNCTIONS

Any function of the form: $f(x) = ax^b$ where a and b are real numbers not equal to zero.

***Exercise* #1:** For each of the following power functions, state the value of a and b by writing the equation in the form $y = ax^b$.

(a) $y = \dfrac{3}{x^2}$ (b) $y = \dfrac{1}{7x^3}$ (c) $y = 8\sqrt{x}$ (d) $y = \dfrac{6}{\sqrt[3]{x}}$

The characteristics of power functions depend on both the value of a and the value of b. The most important, though, is the exponent (the a is simply a vertical stretch of the power function).

***Exercise* #2:** Consider the general power function $y = ax^b$.

(a) What can be said about the y-intercept of any power function if $b > 0$? Illustrate.

(b) What can be said about the y-intercept of any power function if $b < 0$? Illustrate.

For now we will just concentrate on power function where the exponent is a positive whole number.

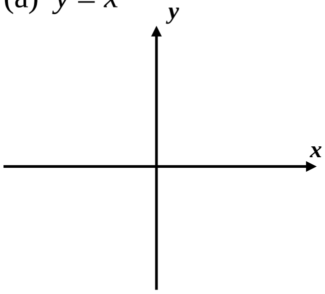
***Exercise* #3:** Using your table, fill in the following values for common power functions.

x	-3	-2	-1	0	1	2	3
x^2							
x^3							
x^4							
x^5							

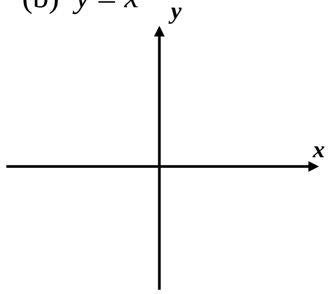
From the previous exercise, we should note that when the power function has an even exponent, then positive and negative INPUTS have the same value. When the power function has an odd exponent, then positive and negative inputs have opposite outputs. Recall this is the definition of **even and odd functions**.

***Exercise* #4:** Using your calculators, sketch the power functions below using the standard viewing window.

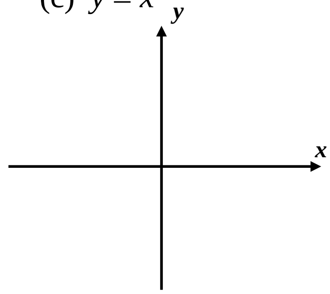
(a) $y = x^2$



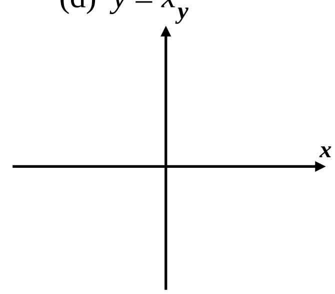
(b) $y = x^3$



(c) $y = x^4$



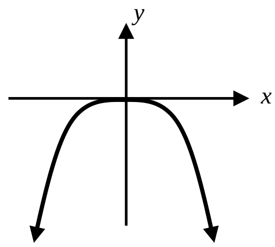
(d) $y = x^5$



***Exercise* #5:** Which of the following power functions is shown in the graph below? Explain your choice. Do without the use of your calculator.

(1) $y = -4x^7$ (3) $y = 6x^8$

(2) $y = -3x^{10}$ (4) $y = 5x^9$



The End Behavior of Polynomials – The behavior of polynomials as the input variable gets very large, both positive and negative, is important to understand. We will explore this in the next exercise.

***Exercise* #6:** Consider the two functions $y_1 = x^3 - 2x^2 - 29x + 30$ and $y_2 = x^3$.

(a) Graph these functions using $x_{\min} = -10,\ x_{\max} = 10,\ y_{\min} = -100,\ y_{\max} = 100$

(b) Graph these functions using $x_{\min} = -20,\ x_{\max} = 20,\ y_{\min} = -1000,\ y_{\max} = 1000$

(c) Graph these functions using $x_{\min} = -50,\ x_{\max} = 50,\ y_{\min} = -10000,\ y_{\max} = 10000$

(d) Graph these functions using $x_{\min} = -100,\ x_{\max} = 100,\ y_{\min} = -100000,\ y_{\max} = 100000$

(e) What do you observe about the nature of the two graphs as the viewing window gets larger?

(f) Why is this occurring?

The end behavior (also known as long-run) of any polynomial is dictated by its highest powered term!!!

3-2 Creating Polynomial Equations

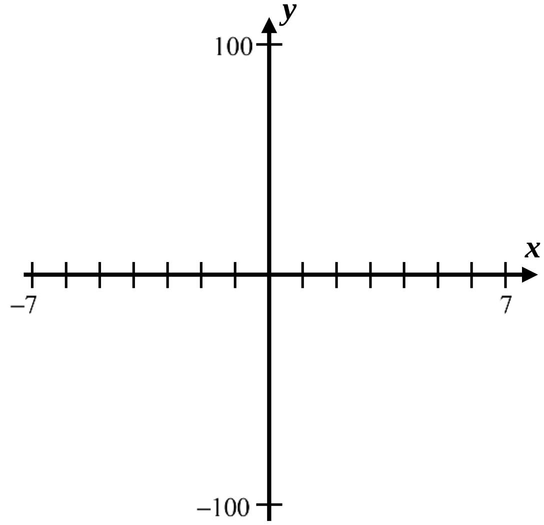
The connection between the zeroes of a polynomial and its factors should now be clear. This connection can be used to create equations of polynomials. The key is utilizing the factored form of a polynomial.

The Factored Form of a Polynomial

If the set $\{r_1, r_2, r_3, \ldots, r_n\}$ represent the roots (zeroes) of a polynomial, then the polynomial can be written as:

$$y = a(x - r_1)(x - r_2)\cdots(x - r_n)$$ where a is some constant determined by another point

***Exercise* #1:** Determine the equation of a quadratic function whose roots are -3 and 4 and which passes through the point $(2, -50)$. Express your answer in standard form ($y = ax^2 + bx + c$). Verify your answer by creating a sketch of the function on the axes below.



It's important to understand how the a value effects the graph of the polynomial. This is easiest to explore if the polynomial remains in factored form.

***Exercise* #2:** Consider quadratic polynomials of the form $y = a(x+2)(x-5)$, where $a \neq 0$.

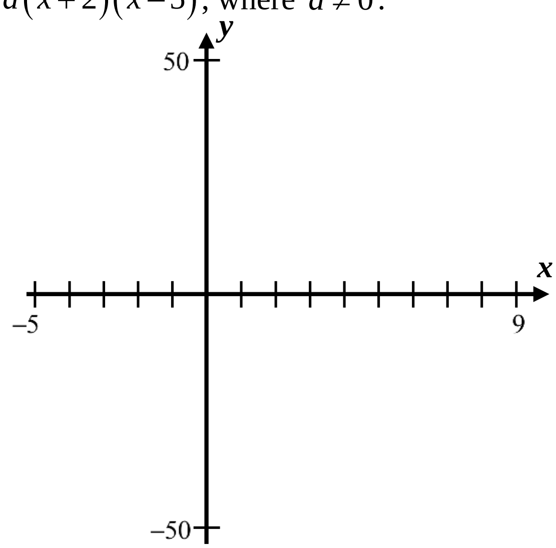
(a) What are the x-intercepts of this parabola?

(b) Sketch on the axes given the following equations:

$y = (x+2)(x-5)$

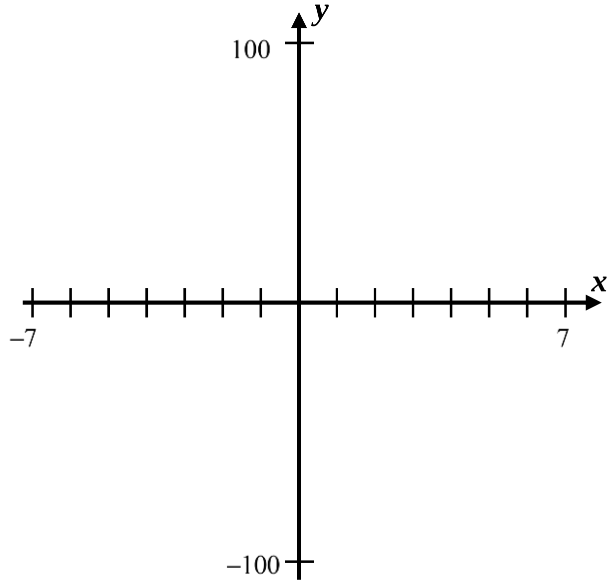
$y = 2(x+2)(x-5)$

$y = 4(x+2)(x-5)$

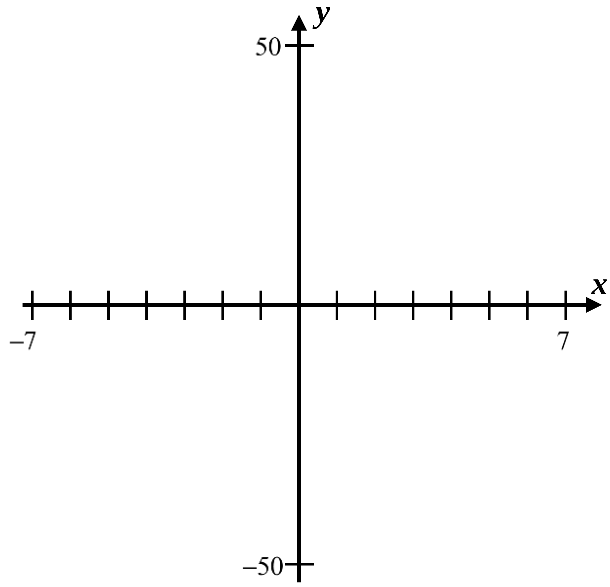


As we can see from this exercise, the value of a does not change the zeroes of the function, but does vertically stretch the function. We can create equations of higher powered polynomials in a similar fashion.

***Exercise* #3:** Create the equation of the cubic, in standard form, that has x-intercepts of $-4, 2,$ and 5 and passes through the point $(6, 20)$. Verify your answer by sketching the cubic's graph on the axes below.



***Exercise* #4:** Create the equation of a cubic in standard form that has a double zero at -2 and another zero at 4. The cubic has a y-intercept of 16. Sketch your cubic on the axes below to verify your result.



***Exercise* #5:** How would you describe this cubic curve at its double root?

3-3 GRAPHS AND ZEROES OF A POLYNOMIAL

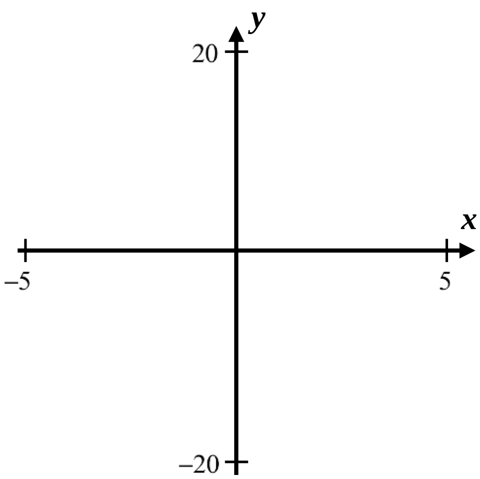
A polynomial is a function consisting of terms that all have whole number powers. In its most general form, a polynomial can be written as:

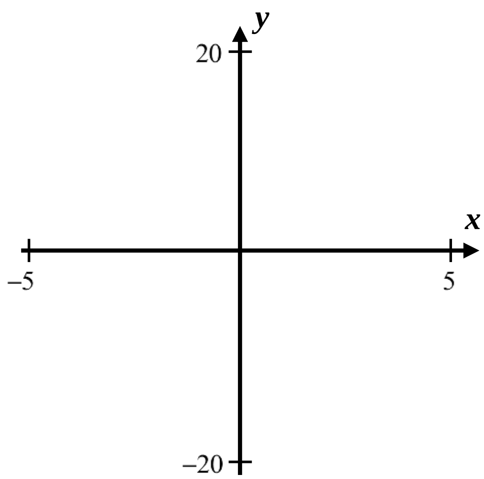
$$y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

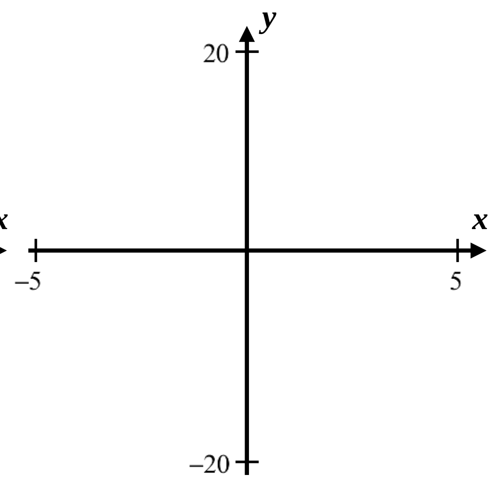
Quadratic and linear functions are the simplest of all polynomials. In this lesson we will explore cubic and quartic functions, those whose highest powers are x^3 and x^4 respectively.

***Exercise* #1:** For each of the following cubic functions, sketch the graph and circle its *x*-intercepts.

(a) $y = x^3 - 3x^2 - 6x + 8$ (b) $y = 2x^3 - 8x + 9$ (c) $y = 2x^3 - 12x^2 + 18x$





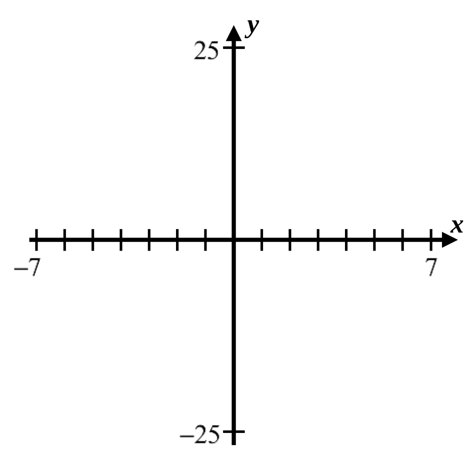


Clearly, a cubic may have one, two or three real roots and can have two turning points. Just as with parabolas, there exists a tie between a cubic's factors and its *x*-intercepts.

***Exercise* #2:** Consider the cubic whose equation is $y = x^3 - x^2 - 12x$.

(a) *Algebraically* determine the zeroes of this function.

(b) Sketch a graph of this function on the axes below illustrating your answer to part (a).



***Exercise* #3:** The largest root of $x^3 - 9x^2 + 12x + 22 = 0$ falls between what two consecutive integers?

(1) 4 and 5 (3) 10 and 11

(2) 6 and 7 (4) 8 and 9

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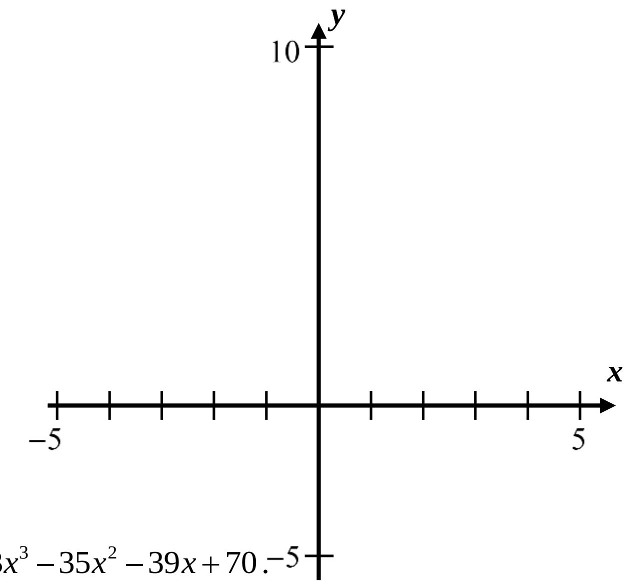
***Exercise* #4:** Consider the quartic function $y = x^4 - 5x^2 + 4$.

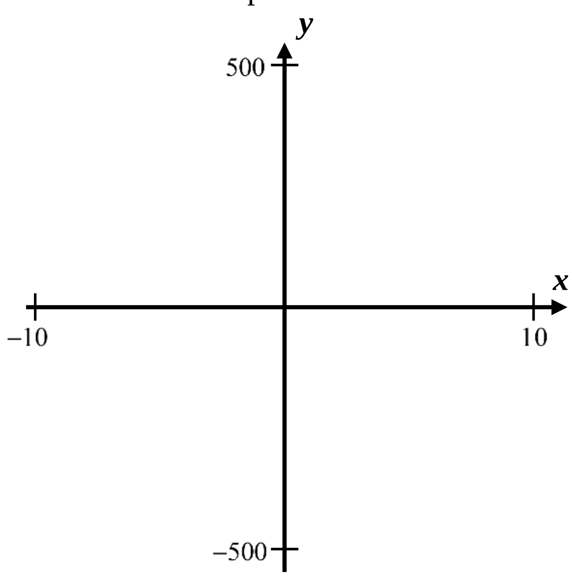
(a) *Algebraically* determine the x-intercepts of this function.

(b) Verify your answer to part (a) by sketching a graph of the function on the axes below.



***Exercise* #5:** Consider the quartic whose equation is $y = x^4 + 3x^3 - 35x^2 - 39x + 70$.

(a) Sketch a graph of this quartic on the axes below. Label its x-intercepts.



(b) Based on your graph from part (a), write the expression $x^4 + 3x^3 - 35x^2 - 39x + 70$ in its factored form.

3-4 POLYNOMIAL LONG DIVISION

We have worked to simplify rational expressions (polynomials divided by polynomials). In this lesson, we will look more closely at the division of two polynomials and how it is analogous to the division of two integers.

***Exercise* #1:** Consider the division problem $1519 \div 7$, which could also be written as $\frac{1519}{7}$ and $7\overline{)1519}$.

(a) Find the result of this division using the standard long division algorithm. Is there a remainder in this division?

(b) Rewrite your result from (a) as an equivalent multiplication equation.

(c) Now evaluate $\frac{1522}{7}$ using long division. Write your answer in $a+\text{R}b$ form and in $a+\frac{b}{c}$ form,

(d) Write your answer from part (c) as an equivalent multiplication equation.

***Exercise* #2:** Now let's see how this works out when we divide two polynomials.

(a) Simplify $\frac{2x^2+15x+18}{x+6}$ by performing polynomial long division.

(b) Rewrite the result you found in (a) as an equivalent multiplication equation.

(c) Write $\frac{2x^2+15x+20}{x+6}$ in the form $q(x)+\frac{r}{(x+6)}$, where $q(x)$ is a polynomial and r is a constant, by performing polynomial long division. Also, write the result an equivalent multiplication equation.

So, when we divide two polynomials, we always get another polynomial and a remainder. This is known as writing the rational expression in **quotient-remainder form.**

***Exercise* #3:** Write each of the following rational expressions in the form $q(x)+\dfrac{r}{(x-a)}$ form.

(a) $\dfrac{x^2+2x-5}{x-3}$

(b) $\dfrac{2x^2-23x+17}{x-10}$

Sometimes we can use the structure of an expression instead of polynomial long division.

***Exercise* #4:** Consider the expression $\dfrac{x+8}{x+3}$. We would like to write this as $a+\dfrac{b}{x+3}$.

(a) Write the numerator as an equivalent expression involving the expression $x+3$.

(b) Use the fact that division distributes over addition to write the final answer.

We can extend what we did in the last problem to more challenging structure problems.

***Exercise* #5:** Write each of the following in the form of $a+\dfrac{b}{x-r}$.

(a) $\dfrac{4x+13}{x+2}$

(b) $\dfrac{3x-5}{x-4}$

3-5 The Remainder Theorem

In the last lesson, we saw how two polynomials, when divided, resulted in another polynomial and a remainder. The remainder has a remarkable property in certain types of division. We will explore this relationship in the first exercise.

***Exercise* #1:** Consider each of the following scenarios where we have $\frac{p(x)}{x-a}$. In each case, simplify the division using polynomial long division and then evaluate $p(a)$.

(a) $\frac{x^2-8x+18}{x-2}$ $\qquad$ $p(x)=x^2-8x+18 \Rightarrow p(2)=$

(b) $\frac{x^2-2x-25}{x-7}$ $\qquad$ $p(x)=x^2-2x-25 \Rightarrow p(7)=$

(c) $\frac{2x^2+11x+11}{x+3}$ $\qquad$ $p(x)=2x^2+11x+11 \Rightarrow p(-3)=$

(d) $\frac{3x^2+7x-20}{x+4}$ $\qquad$ $p(x)=3x^2+7x-20 \Rightarrow p(-4)=$

THE REMAINDER THEOREM

When the polynomial $p(x)$ is divided by the binomial $(x-a)$ then the remainder will always be $p(a)$. In other words:

$$\frac{p(x)}{(x-a)} = q(x) + \frac{p(a)}{x-a}$$

***Exercise* #2:** If the ratio $\frac{x^2-11x+22}{x-9}$ was placed in the form $q(x)+\frac{r}{x-9}$, where $q(x)$ is a linear function, then which of the following is the value of *r*?

(1) -3 (3) -9

(2) 5 (4) 4

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In the past, the remainder theorem was used primarily to aid in evaluating polynomials. These days it is the primary justification for telling if a linear expression is a factor of a polynomial.

***Exercise* #3:** By definition $(x-a)$ is a factor of $p(x)$ if $\frac{p(x)}{x-a}=q(x)$, where $q(x)$ is another polynomial. What must be true of the remainder, $p(a)$, for $(x-a)$ to be a factor of $p(x)$? Explain.

***Exercise* #4:** Determine if each of the following are factors of the listed polynomials by evaluating the polynomials.

(a) Is $x-3$ a factor of $p(x)=x^2-11x+24$?

(b) Is $x+5$ a factor of $p(x)=2x^2+9x-2$?

(d) Is $x+1$ a factor of $p(x)=x^3-7x^2-11x-3$

(c) Is $x-5$ a factor of $p(x)=x^3-x^2-19x-10$?

***Exercise* #5:** For what value of *k* will $x-4$ be a factor of $x^2+kx-52$? Show how you arrived at your answer.