Name: ______________________________________ Date: ___________________

5-1 EXPONENTIAL FUNCTION BASICS

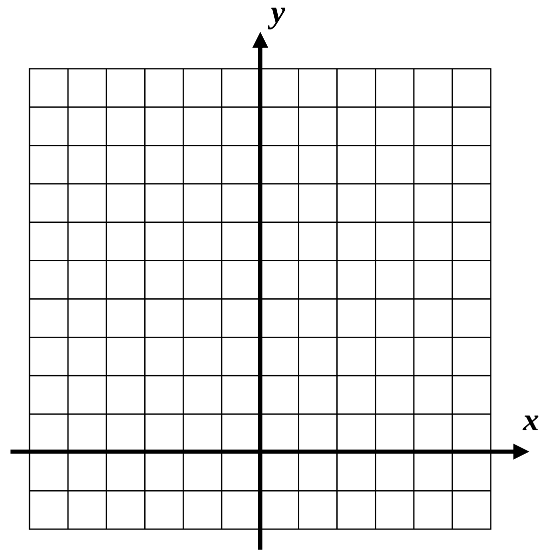
You studied exponential functions extensively in Common Core Algebra I. Today's lesson will review many of the basic components of their graphs and behavior. Exponential functions, those whose exponents are variable, are extremely important in mathematics, science, and engineering.

BASIC EXPONENTIAL FUNCTIONS

$y = b^x$ where $b > 0$ and $b \neq 1$

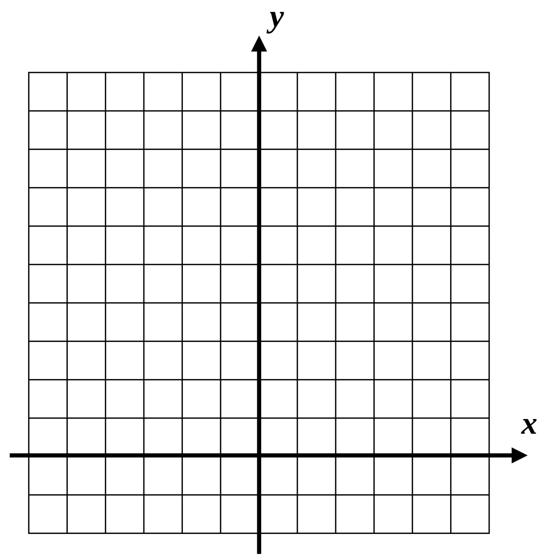
***Exercise* #1:** Consider the function $y = 2^x$. Fill in the table below without using your calculator and then sketch the graph on the grid provided.

x	$y = 2^x$
–3	
–2	
–1	
0	
1	
2	
3	



***Exercise* #2:** Now consider the function $y = \left(\frac{1}{2}\right)^x$. Using your calculator to help you, fill out the table below and sketch the graph on the axes provided.

x	$y = \left(\frac{1}{2}\right)^x$
–3	
–2	
–1	
0	
1	
2	
3	

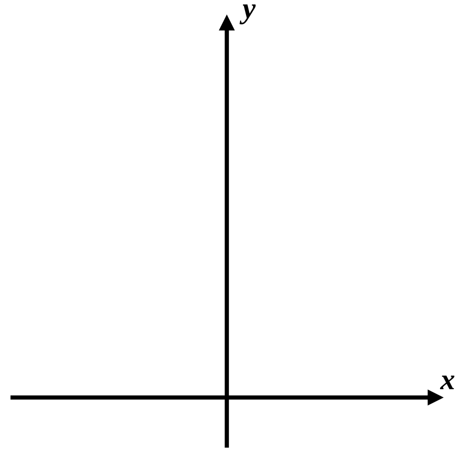


***Exercise* #3:** Based on the graphs and behavior you saw in *Exercises* #1 and #2, state the domain and range for an exponential function of the form $y = b^x$.

Domain (input set): Range (output set):

***Exercise* #4:** Are exponential functions one-to-one? How can you tell? What does this tell you about their inverses?

***Exercise* #5:** Now consider the function $y = 7(3)^x$.

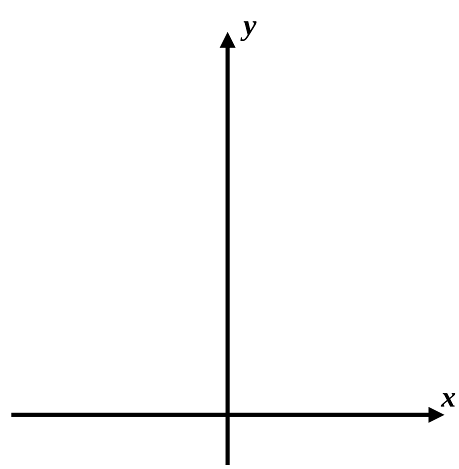


(a) Determine the *y*-intercept of this function algebraically. Justify your answer.

(b) Does the exponential function increase or decrease? Explain your choice.

(c) Create a rough sketch of this function, labeling its *y*-intercept.

***Exercise* #6:** Consider the function $y = \left(\frac{1}{3}\right)^x + 4$.



(a) How does this function's graph compare to that of $y = \left(\frac{1}{3}\right)^x$? What does adding 4 do to a function's graph?

(b) Determine this graph's *y*-intercept algebraically. Justify your answer.

(c) Create a rough sketch of this function, labeling its *y*-intercept.

5-2 EXPONENTIAL MODELING WITH PERCENT GROWTH AND DECAY

Exponential functions are very important in modeling a variety of real world phenomena because certain things either increase or decrease by **fixed percentages** over given units of time. You looked at this in Common Core Algebra I and in this lesson we will review much of what you saw.

***Exercise* #1:** Suppose that you deposit money into a savings account that receives 5% interest per year on the amount of money that is in the account for that year. Assume that you deposit $400 into the account initially.

(a) How much will the savings account increase by over the course of the year?

(b) How much money is in the account at the end of the year?

(c) By what single number could you have multiplied the $400 by in order to calculate your answer in part (b)?

(d) Using your answer from part (c), determine the amount of money in the account after 2 and 10 years. Round all answers to the nearest cent when needed.

(e) Give an equation for the amount in the savings account $S(t)$ as a function of the number of years since the $400 was invested.

(f) Using a table on your calculator determine, to the nearest year, how long it will take for the initial investment of $400 to double. Provide evidence to support your answer.

The thinking process from *Exercise* #1 can be generalized to any situation where a quantity is increased by a fixed percentage over a fixed interval of time. This pattern is summarized below:

INCREASING EXPONENTIAL MODELS

If quantity *Q* is known to increase by a fixed percentage *p*, in decimal form, then *Q* can be modeled by

$$Q(t) = Q_0(1+p)^t$$

where Q_0 represents the amount of *Q* present at $t = 0$ and *t* represents time.

***Exercise* #2:** Which of the following gives the savings *S* in an account if $250 was invested at an interest rate of 3% per year?

(1) $S = 250(4)^t$

(2) $S = 250(1.03)^t$

(3) $S = (1.03)^t + 250$

(4) $S = 250(1.3)^t$

Decreasing exponentials are developed in the same way, but have the percent subtracted, rather than added, to the base of 100%. Just remember, you are ultimately multiplying by the percent of the original that you will have after the time period elapses.

***Exercise* #3:** State the multiplier (base) you would need to multiply by in order to decrease a quantity by the given percent listed.

(a) 10% (b) 2% (c) 25% (d) 0.5%

DECREASING EXPONENTIAL MODELS

If quantity *Q* is known to decrease by a fixed percentage *p*, in decimal form, then *Q* can be modeled by

$$Q(t) = Q_0(1-p)^t$$

where Q_0 represents the amount of *Q* present at $t = 0$ and *t* represents time.

***Exercise* #4:** If the population of a town is decreasing by 4% per year and started with 12,500 residents, which of the following is its projected population in 10 years? Show the exponential model you use to solve this problem.

(1) 9,230 (3) 18,503

(2) 76 (4) 8,310

***Exercise* #5:** The stock price of WindpowerInc is increasing at a rate of 4% per week. Its initial value was $20 per share. On the other hand, the stock price in GerbilEnergy is crashing (losing value) at a rate of 11% per week. If its price was $120 per share when Windpower was at $20, after how many weeks will the stock prices be the same? Model both stock prices using exponential functions. Then, find when the stock prices will be equal graphically. Draw a well labeled graph to justify your solution.

5-2 COMPOUND INTEREST

In the worlds of investment and debt, interest is added onto a principal in what is known as **compound interest**. The percent rate is typically given on a yearly basis, but could be applied more than once a year. This is known as the **compounding frequency**. Let's take a look at a typical problem to understand how the compounding frequency changes how interest is applied.

***Exercise* #1:** A person invests $500 in an account that earns a **nominal yearly interest rate** of 4%.

(a) How much would this investment be worth in 10 years if the **compounding frequency** was once per year? Show the calculation you use.

(b) If, on the other hand, the interest was applied four times per year (known as quarterly compounding), why would it not make sense to multiply by 1.04 each quarter?

(c) If you were told that an investment earned 4% per year, how much would you assume was earned per quarter? Why?

(d) Using your answer from part (c), calculate how much the investment would be worth after 10 years of quarterly compounding? Show your calculation.

So, the pattern is fairly straightforward. For a **shorter compounding period**, we get to **apply the interest more often**, but at a **lower rate**.

***Exercise* #2:** How much would $1000 invested at a nominal 2% yearly rate, compounded monthly, be worth in 20 years? Show the calculations that lead to your answer.

(1) $1485.95 (3) $1033.87

(2) $1491.33 (4) $1045.32

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This pattern is formalized in a classic formula from economics that we will look at in the next exercise.

***Exercise* #3:** For an investment with the following parameters, write a formula for the amount the investment is worth, *A*, after *t*-years.

P = amount initially invested

r = nominal yearly rate

n = number of compounds per year

$A(t) =$

The rate in *Exercise* #1 was referred to as **nominal** (**in name only)**. It's known as this, because you effectively earn more than this rate if the compounding period is more than once per year. Because of this, bankers refer to the **effective rate**, or the rate you would receive if compounded just once per year. Let's investigate this.

***Exercise* #4:** An investment with a nominal rate of 5% is compounded at different frequencies. Give the **effective** yearly rate, accurate to two decimal places, for each of the following compounding frequencies. Show your calculation.

(a) Quarterly

(b) Monthly

(c) Daily

<u>Practice:</u> 1. You deposit $10,000 in an account that pays 6% interest. Find the balance after 10 years if the interest is compounded

a) quarterly

b) Monthly

2. $2000 is deposited in an account that pays 8% annual interest, compounded monthly. What is the balance after 5 years?

3. A parent, following the birth of a child, wants to make an initial investment that will grow to $10,000 by the child's 20th birthday. Interest is compounded continuously at 8%. What should that initial investment be?

4. Complete the table: Invest $1 for 1 year at 100% compound interest and compare the result.

Annually	Bi-Annually	Quarterly	Monthly	Weekly	Daily	Hourly	Every Minute	Every Second

Conclusion:

Quick write: Write down any symbols that have defined values. Why are these symbols used instead of numbers?

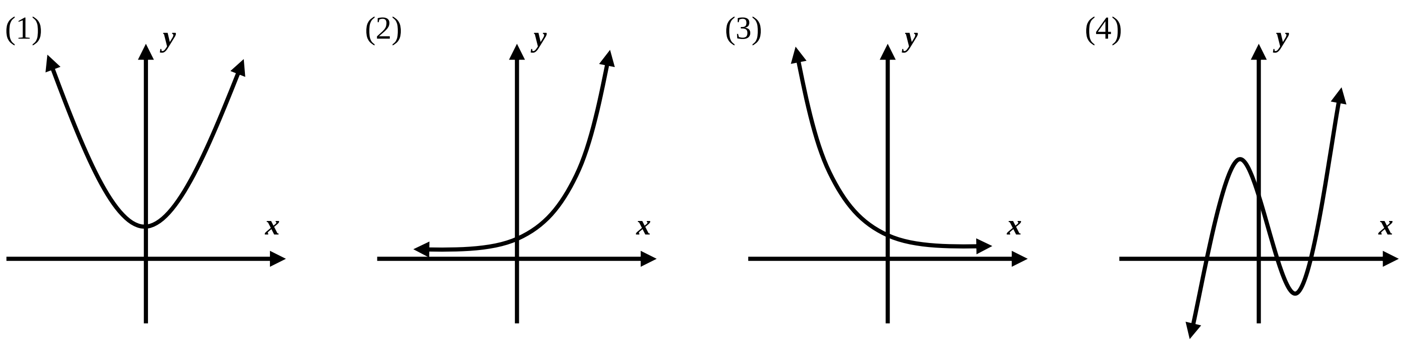
5-3 THE NUMBER e AND THE NATURAL LOGARITHM

There are many numbers in mathematics that are more important than others because they find so many uses in either mathematics or science. Good examples of important numbers are 0, 1, *i*, and π. In this lesson you will be introduced to an important number given the letter **e** for its "inventor" Leonhard Euler (1707-1783). This number plays a crucial role in Calculus and more generally in modeling exponential phenomena.

THE NUMBER e

1. Like π, e is irrational. 2. $e \approx 2.72$ 3. Used in Exponential Modeling

***Exercise* #1:** Which of the graphs below shows $y = e^x$? Explain your choice. Check on your calculator.



Explanation:

Very often e is involved in exponential modeling of both increasing and decreasing quantities. The creation of these models is beyond the scope of this course, but we can still work with them.We could compound at smaller and smaller frequency intervals, eventually compounding all moments of time. In our formula from *Exercise* #3, we would be letting *n* approach infinity. Interestingly enough, this gives rise to **continuous compounding** and the use of the natural base **e** in the famous **continuous compound interest formula.**

CONTINUOUS COMPOUND INTEREST

For an initial principal, *P*, compounded continuously at a nominal yearly rate of *r*, the investment would be worth an amount *A* given by:

$$A(t) = Pe^{rt}$$

***Exercise* #5:** A person invests \$350 in a bank account that promises a nominal rate of 2% continuously compounded.

(a) Write an equation for the amount this investment would be worth after *t*-years.

(b) How much would the investment be worth after 20 years?

(c) Algebraically determine the time it will take for the investment to reach \$400. Round to the nearest tenth of a year.

(d) What is the effective annual rate for this investment? Round to the nearest hundredth of a percent.

Practice: A student wants to save \$8,000 for college in four years. How much should be put into an account that earns 5.2% annual interest compounded continuously?

5. How long would it take to double your principal at an annual interest rate of 8% compounded continuously?