Name: **Answer Key** Date: ____________

UNIT #5 – EXPONENTIAL AND LOGARITHMIC FUNCTIONS
MATH III HONORS

Part I Questions

1. The expression $\left(\frac{1}{x^3}\right)^2$ is equivalent to

(1) x^{-1} (3) x^{-5}

(2) $x^{2/3}$ (4) x^{-6}

$$=\frac{1^2}{\left(x^3\right)^2}=\frac{1}{x^{3\cdot 2}}=\frac{1}{x^6}=x^{-6}$$

(4)

2. The exponential function $y=16\left(2^x\right)$ could be rewritten as

(1) $y=32^x$ (3) $y=2^{x+4}$

(2) $y=2^{5x}$ (4) $y=2^{x^3}$

$$y=16\left(2^x\right)=2^4\cdot 2^x=2^{4+x}=2^{x+4}$$

(3)

3. The expression $a^{\frac{5}{2}}$ is equivalent to which of the following as long as $a>0$?

(1) $\sqrt{a^5}$ (3) $\sqrt[5]{a^2}$

(2) $\sqrt{5a}$ (4) $\frac{5a}{2}$

$$a^{5/2}=a^{5\cdot 1/2}=\left(a^5\right)^{1/2}=\sqrt{a^5}$$

(1)

4. Which of the following would give the same result as $\left(\sqrt{\sqrt[3]{2}}\right)^4$?

(1) $\sqrt[5]{8}$ (3) $\sqrt{2}$

(2) $\sqrt[4]{8}$ (4) $\sqrt[3]{4}$

$$\left(\left(2^{1/3}\right)^{1/2}\right)^4=2^{1/3\cdot 1/2\cdot 4}=2^{4/6}=2^{2/3}=\sqrt[3]{2^2}=\sqrt[3]{4}$$

(4)

5. For the function $f(x)=5(2)^x+7$, which of the following represents its y-intercept?

(1) 7 (3) 12

(2) 5 (4) 17

$$\begin{aligned} f(0)&=5(2)^0+7\\ &=5(1)+7\\ &=5+7\\ &=12 \end{aligned}$$

(3)

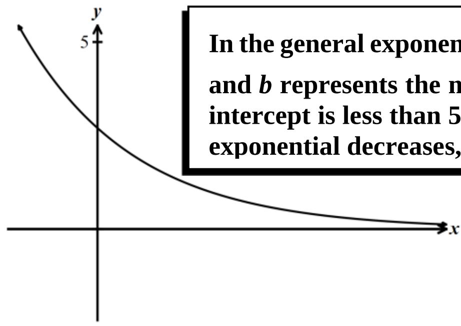
6. Which of the following could be the equation of the graph shown below?

(1) $y = 10(0.5)^x$

(2) $y = 3(0.75)^x$

(3) $y = 4(1.25)^x$

(4) $y = 5(2.2)^x$



In the general exponential equation $y = a(b)^x$, *a* represents the *y*-intercept and *b* represents the multiplicative growth or decay constant. Since the *y*-intercept is less than 5, we could only have choice (2) or (3). But, since the exponential decreases, $0 < b < 1$, thus it must be choice (2).

(2)

7. Which of the following values of *x* solves: $(0.5)^{3x+2} = 8^{5x-4}$?

(1) $\frac{2}{3}$ (3) 3

(2) $\frac{5}{9}$ (4) 7

$$\left(2^{-1}\right)^{3x+2} = \left(2^3\right)^{5x-4}$$
$$2^{-3x-2} = 2^{15x-12}$$
$$-3x - 2 = 15x - 12$$
$$18x = 10$$
$$x = \frac{10}{18} = \frac{5}{9}$$

(2)

8. A population of fruit flies is increasing at a rate of 22.5% per hour. If the population had an original size of 10 flies, then which of the following is its size after one day?

(1) 798 (3) 1122

(2) 935 (4) 1304

$$10(1.225)^{24} = 1303.96... = 1304$$

(4)

9. The water level in a draining reservoir is changing such that the depth of water decreases by 7.5% per hour. If the water starts at a depth of 45 feet, then which of the following functions properly models the depth, *d*, as a function of time, *t*, in hours since it started draining?

(1) $d = 45(.075)^t$ (3) $d = 45(7.5)^t$

(2) $d = 45(.925)^t$ (4) $d = 45(92.5)^t$

100% – 7.5% = 92.5% of the water remains

$$d = 45(0.925)^t$$

(2)

10. The temperature of a cooling liquid in a room held at a constant 75 degrees Fahrenheit can be described by the equation $F(t) = 132(.97)^t + 75$, where *F* is the Fahrenheit temperature and *t* is the amount of time it has been cooling, in minutes. Which of the following was the original temperature of the liquid when it began cooling?

(1) 75 (3) 203

(2) 132 (4) 207

$$F(0) = 132(.97)^0 + 75$$
$$= 132(1) + 75$$
$$= 132 + 75$$
$$= 207$$

(4)

11. If a population grows at a constant rate of 2.8% per year, then by what percent will it grow over the next 10 years?

(1) 17% (3) 32%

(2) 28% (4) 39%

$$\left(1.028\right)^{10} = 1.31804... \approx 1.32$$
$$= 1 + 0.32$$

Thus, since we are multipying by 1.32 per 10 years, the 10 year increase is approximately 32%.

(3)

12. The half-life of a radioactive material is the amount of time it takes for 50% of its radioactivity to decrease. If a particular material has a half-life of 35 years, then what percent will remain radioactive after 100 years?

(1) 13.8% (3) 34.8%

(2) 22.7% (4) 48.7%

Since we multiply by 0.5 every 35 years, this means we multiply by $(0.5)^{1/35}$ every year. So, if we apply this for 100 years we get:

$$\left(\left(0.5\right)^{1/35}\right)^{100} = 0.13801 \Rightarrow 13.8\%$$

(1)

13. Which of the following is closest to the value of $\log_4\left(40\right)$?

(1) 1.8 (3) 2.7

(2) 2.3 (4) 3.5

$4^{1.8} = 12.125$

$4^{2.3} = 24.25$

$4^{2.7} = 42.22$ **(closest to 40)**

$4^{3.5} = 128$

(3)

14. If $b > 0$ then $\log_b\left(\frac{1}{b^3}\right)$ is equal to

(1) $\frac{1}{3}$ (3) 3

(2) $\frac{b}{3}$ (4) -3

$$\log_b\left(\frac{1}{b^3}\right) = \log_b\left(b^{-3}\right) = -3$$

(4)

15. Given the function $f\left(x\right) = \log_2\left(2x-8\right)$, which of the following values of x is *not* in the domain of the function?

(1) $x = 5$ (3) $x = 8$

(2) $x = 2$ (4) $x = 20$

The argument of the logarithm, $2x-8$, must be greater than zero.

$$2x - 8 > 0 \Rightarrow 2x > 8 \Rightarrow x > 4$$

(2)

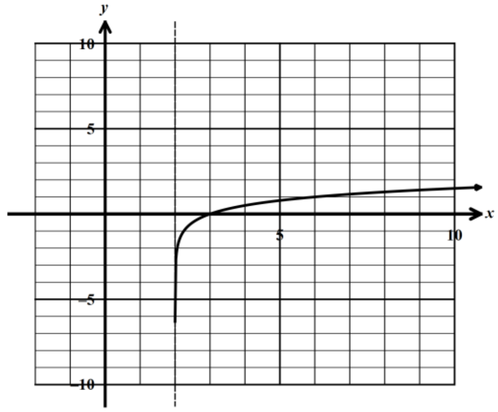
16. Which of the following equations is shown graphed on the grid below?

(1) $y = 2^x + 2$

(2) $y = 4^x - 2$

(3) $y = \log_4(x-2)$

(4) $y = \log_2(x+2)$



This is a standard logarithm graph, $y = \log_b(x)$ shifted 2 units to the right. Thus, it must be:

$$y = \log_4(x-2)$$

(3)

17. Which of the following is equivalent to $\log\left(\frac{x^2}{\sqrt{y}}\right)$?

(1) $(\log x)^2 - \sqrt{\log y}$

(2) $\frac{2\log x}{\sqrt{\log y}}$

(3) $\frac{2\log x - \log y}{2}$

(4) $\frac{4\log x - \log y}{2}$

$$\log\left(\frac{x^2}{\sqrt{y}}\right) = \log(x^2) - \log(\sqrt{y})$$
$$= \log(x^2) - \log\left(y^{1/2}\right)$$
$$= 2\log x - \frac{1}{2}\log(y)$$
$$= \frac{4\log x}{2} - \frac{\log y}{2} = \frac{4\log x - \log y}{2}$$

(4)

18. If $\log_b(5) = 1.2$ then $\log_b(125) =$?

(1) 0.4

(2) 1.728

(3) 3.6

(4) 30

$$\log_b(125) = \log_b(5^3)$$
$$= 3\log_b(5)$$
$$= 3(1.2)$$
$$= 3.6$$

(3)

19. If $5b^{x-3} = 7$ then $x =$

(1) $\frac{\log_b(7)}{5} + 3$

(2) $\frac{5b}{7} - 3$

(3) $3 + \log_b(1.4)$

(4) $3b^{7/5}$

$$5b^{x-3} = 7$$
$$b^{x-3} = \frac{7}{5}$$
$$\log_b\left(b^{x-3}\right) = \log_b\left(\frac{7}{5}\right) = \log_b(1.4)$$
$$x - 3 = \log_b(1.4)$$
$$x = 3 + \log_b(1.4)$$

(3)

20. If $f(x) = 50(0.92)^x + 75$ then which of the following values of x solves the equation $f(x) = 90$?

(1) 12.1

(2) 14.4

(3) 15.8

(4) 18.3

$$50(0.92)^x + 75 = 90 \Rightarrow 50(0.92)^x = 15 \Rightarrow 0.92^x = \frac{15}{50} = 0.3$$
$$\log\left(0.92^x\right) = \log(0.3) \Rightarrow x\log(0.92) = \log(0.3)$$
$$x = \frac{\log(0.3)}{\log(0.92)} = 14.4$$

(2)

21. If $ae^{kt} - c = 0$ then which of the following is the value of *t* based on *a*, *k*, and *c* and the natural base e?

(1) $\frac{1}{k}\ln\left(\frac{c}{a}\right)$ (3) $\ln\left(\frac{c}{ak}\right)$

(2) $\frac{\ln(c)}{ak}$ (4) $\frac{ac}{ke}$

$$ae^{kt} = c$$
$$e^{kt} = \frac{c}{a}$$
$$\ln\left(e^{kt}\right) = \ln\left(\frac{c}{a}\right)$$

$$\Rightarrow$$

$$\cancel{\ln}\left(\cancel{e}^{kt}\right) = \ln\left(\frac{c}{a}\right)$$
$$kt = \ln\left(\frac{c}{a}\right)$$
$$t = \frac{1}{k}\ln\left(\frac{c}{a}\right)$$

(1)

22. If \$500 is placed in a savings account that earns a 6% nominal interest compounded monthly, then which of the following represents the account's worth after 10 years?

(1) \$800.00 (3) \$895.42

(2) \$873.29 (4) \$909.70

$$500\left(1+\frac{.06}{12}\right)^{10\cdot 12} = 500\left(1+\frac{.06}{12}\right)^{120}$$
$$= \$909.70$$

(4)

23. How many years, to the nearest tenth, would it take for an investment to double if it is earning a continuous compound interest of 3.5% per year?

(1) 17.4 years (3) 22.5 years

(2) 19.8 years (4) 25.1 years

$$A = Pe^{rt}$$
$$A = Pe^{.035t}$$
$$A = 2P \Rightarrow t = ?$$

$$\Rightarrow$$

$$Pe^{.035t} = 2P$$
$$e^{.035t} = 2$$
$$\ln\left(e^{.035t}\right) = \ln 2$$
$$.035t = \ln 2$$
$$t = \frac{\ln 2}{.035} = 19.8$$

(2)

24. If a liquid is cooling down according to the formula $y = 84e^{kt} + 55$ and at $t = 22$ the temperature is $y = 71$ then which of the following is the value of *k* to the nearest hundredth?

(1) -0.08 (3) 0.29

(2) -0.27 (4) 0.58

$$84e^{k\cdot 22} + 55 = 71$$
$$84e^{22k} + 55 = 71$$
$$84e^{22k} = 16$$
$$e^{22k} = \frac{16}{84}$$

$$\ln\left(e^{22k}\right) = \ln\left(\frac{16}{84}\right) \Rightarrow 22k = \ln\left(\frac{16}{84}\right)$$
$$k = \frac{\ln\left(\frac{16}{84}\right)}{22} = -.0753... = -.08$$

(1)

25. The temperature of a cooling liquid is given by the function $T(m) = 38(0.82)^m + 21$, where *T* represents the temperature in degrees Celsius and *m* represents the number of minutes, $m \geq 0$, that the liquid has been cooling. Which of the following represents a temperature that the liquid does not reach as it cools down?

(1) 53 (3) 41

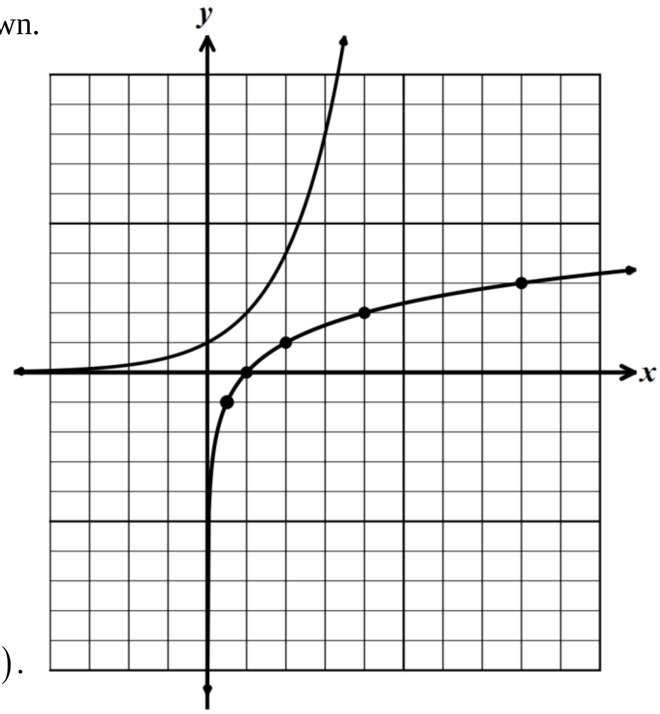
(2) 16 (4) 28

The starting temperature of the liquid is 38+21 or 59 degrees. The room temperature is 21 degrees. The liquid will hit all temperatures between 21 and 59, thus will *not* hit 16.

(2)

Free Response Questions

26. On the grid shown below, the graph of $f(x) = 2^x$ is shown.



(a) On the same graph grid, create an accurate sketch of this function's inverse, $f^{-1}(x)$.

$$\boxed{\begin{array}{l} \left(-1, \frac{1}{2}\right) \to \left(\frac{1}{2}, -1\right) \\ (0, 1) \to (1, 0) \\ (1, 2) \to (2, 1) \\ (2, 4) \to (4, 2) \\ (3, 8) \to (8, 3) \end{array}}$$

(b) State the equation of $f^{-1}(x)$.

$$\boxed{y = \log_2(x)}$$

(c) State the domain and range of both $f(x)$ and $f^{-1}(x)$.

	$f(x)$	$f^{-1}(x)$
Domain:	**All real numbers or ¡**	$x > 0$
Range:	$y > 0$	**All real numbers or ¡**

27. The expression $\left(\sqrt[3]{b}\right)^5 \left(\frac{1}{b^2}\right)$ can be written as b^a in simplest form. Determine the value of a. Show how you arrived at your answer.

$$\boxed{\left(b^{\frac{1}{3}}\right)^5 \left(b^{-2}\right) = \left(b^{\frac{1}{3}\cdot 5}\right)\left(b^{-2}\right) = \left(b^{\frac{5}{3}}\right)\left(b^{-2}\right) = b^{\frac{5}{3}-2} = b^{\frac{5}{3}-\frac{6}{3}} = b^{-\frac{1}{3}}}$$

$$\boxed{a = -\frac{1}{3}}$$

28. If $g(x) = \left(\frac{1}{5}\right)^{2x+7} - 3$ then algebraically determine the solution to the equation $g(x) = 22$.

$$\boxed{\begin{array}{r} \left(\frac{1}{5}\right)^{2x+7} - 3 = 22 \\ \left(\frac{1}{5}\right)^{2x+7} = 25 \\ \left(5^{-1}\right)^{2x+7} = 5^2 \end{array}} \Rightarrow \boxed{\begin{array}{r} 5^{-2x-7} = 5^2 \\ -2x - 7 = 2 \\ -2x = 9 \\ x = -\frac{9}{2} \end{array}}$$

29. For the logarithmic function $f(x)=\log_2(x-4)$, explain why $x=0$ is not in its domain.

$$\begin{aligned} f(0) &= \log_2(0-4) \\ &= \log_2(-4) \end{aligned}$$

It is impossible using real numbers to evaluate this logarithm. A base of 2 raised to any real number will always give a positive result. Thus, there is no output to the expression $\log_2(-4)$ and, hence, 0 is not in the domain.

30. For some base, *b*, it is known that $\log_b(5)=1.28$ and $\log_b(2)=0.55$. For the same base, determine the value of $\log_b(40)$. Explain how you found your answer.

$$\begin{aligned} \log_b(40) &= \log_b(8\cdot 5)=\log_b(8)+\log_b(5) \\ &= \log_b(2^3)+\log_b(5)=3\log_b(2)+\log_b(5) \\ &= 3(0.55)+1.28=2.93 \end{aligned}$$

31. A bank account's worth can be modeled using the formula $w(t)=380\left(1+\frac{.02}{4}\right)^{4t}$, where *w* represents the worth in dollars and *t* represents the number of years since the principal was deposited into the account. Algebraically determine the number of years, to the nearest quarter of a year, it takes for the account to be worth \$500.

$$380\left(1+\frac{.02}{4}\right)^{4t}=500$$
$$\left(1+\frac{.02}{4}\right)^{4t}=\frac{500}{380}=1.3157...$$
$$\log\left(\left(1+\frac{.02}{4}\right)^{4t}\right)=\log(1.3157...)$$

$$4t\log\left(1+\frac{.02}{4}\right)=\log(1.3157...)$$
$$t=\frac{\log(1.3157...)}{4\log\left(1+\frac{.02}{4}\right)}=13.756=13.75$$

Why does it make sense to round your answer to the nearest quarter of a year?

Based on the structure of the function, it is clear that the interest is being applied (compounded) four times a year or quarterly. Thus, it makes sense to round to the nearest quarter of a year.

32. If the population of Ashmore, Illinois is decreasing by 5.8% per year, then by what percent will it decrease in the next 5 years? Show how you arrived at your result. Round to the nearest tenth of a percent.

$100\% - 5.8\% = 94.2\%$ of the population remains ⇨ **$(.942)^5 = .7417... \Rightarrow 74.2\%$ of the population remains after 5 years** ⇨ **$100\% - 74.2\% = 25.8\%$ population decrease**

33. A liquid with an initial temperature of $194\ ^{\circ}\text{F}$ cools in a room whose temperature is held at $68\ ^{\circ}\text{F}$. The temperature of the liquid, *T*, as it cools can be modeled as a function of time, *x*, using:

$$T(x) = (T_i - T_r)e^{kx} + T_r$$

Where T_i is the initial temperature, T_r is the temperature of the room and *k* is the decay constant.

(a) If $T(15) = 102$ then find the value of *k* accurate to the nearest hundredth.

$$\begin{aligned} T(x) &= (194-68)e^{kx} + 68 \\ T(x) &= 126e^{kx} + 68 \end{aligned}$$

⇨

$$\begin{aligned} 102 &= 126e^{k\cdot 15} + 68 \\ 102 &= 126e^{15k} + 68 \\ 34 &= 126e^{15k} \\ e^{15k} &= \frac{34}{126} \end{aligned}$$

⇨

$$\begin{aligned} \ln\left(e^{15k}\right) &= \ln\left(\frac{34}{126}\right) \\ 15k &= \ln\left(\frac{34}{126}\right) \\ k &= \frac{\ln\left(\frac{34}{126}\right)}{15} = -.08732 = -.09 \end{aligned}$$

(b) How many minutes does the model predict it will take for the liquid to reach a temperature of $70\ ^{\circ}\text{F}$? Round to the nearest minute and show or explain how you arrived at your answer.

$$\begin{aligned} T(x) &= 126e^{-.09x} + 68 \\ 70 &= 126e^{-.09x} + 68 \\ 2 &= 126e^{-.09x} \\ e^{-.09x} &= \frac{2}{126} \end{aligned}$$

⇨

$$\begin{aligned} \ln\left(e^{-.09x}\right) &= \ln\left(\frac{2}{126}\right) \\ -.09x &= \ln\left(\frac{2}{126}\right) \\ x &= \frac{\ln\left(\frac{2}{126}\right)}{-.09} = 46 \text{ min} \end{aligned}$$