Name: ___________________________________ Date: _________________

4-0 REVIEW RATIONAL FUNCTIONS

Rational functions are simply the ratio of polynomial functions. They take on more interesting properties and have more interesting graphs than polynomials because of the interaction between the numerator and denominator of the fraction. In Common Core Algebra II, we will be primarily concerned with the algebra of these functions. But in this lesson we will explore some of their characteristics.

***Exercise* #1:** Consider the rational function given by $f(x)=\frac{x+6}{x-3}$.

(a) Algebraically determine the y-intercept for this function.

(b) Algebraically determine the x-intercept of this function. Hint – a fraction can only equal zero if its numerator is zero.

(c) For what value of x is this function undefined? Why is it undefined at this value?

(d) Based on (c), state the domain of this function in set-builder notation.

***Exercise* #2:** Find all values of x for which the rational function $h(x)=\frac{x+5}{2x^2+11x-6}$ is undefined. Verify by using your calculator to evaluate this expression for these values.

***Exercise* #3:** Which of the following represents the domain of the function $f(x)=\frac{x-3}{x^2-6x-16}$?

(1) $\{x \mid x \neq \pm 4\}$

(2) $\{x \mid x \neq 3\}$

(3) $\{x \mid x \neq -2 \text{ and } 8\}$

(4) $\{x \mid x \neq -6 \text{ and } 3\}$

____

***Exercise* #4:** If $g(x) = 3x - 2$ and $f(x) = \dfrac{2x+1}{x+5}$ then find:

(a) $f(g(-2))$

(b) $f(g(2))$

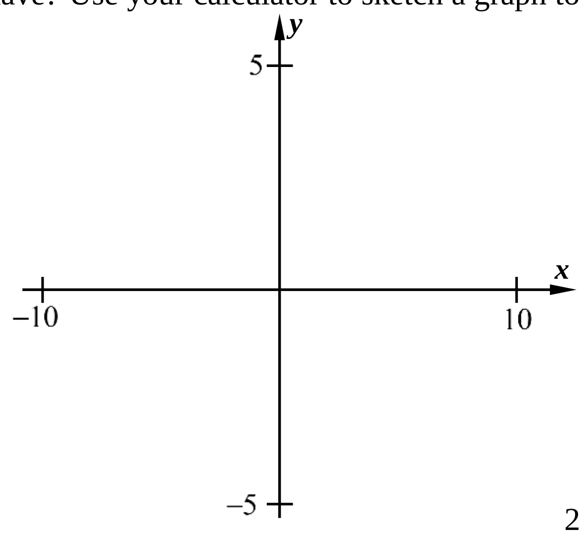
(c) $f(g(x))$

***Exercise* #5:** Find formulas for the inverse of each of the following simple rational functions below. Recall that as a first step, switch the roles of x and y.

(a) $y = \dfrac{x}{x-2}$

(b) $y = \dfrac{x+3}{2x}$

***Exercise* #6:** The function $f(x) = \dfrac{x^2 - 8}{4x}$ is either an even or an odd function. Determine which it is and justify. Based on your answer, what type of symmetry must this function have? Use your calculator to sketch a graph to verify.



4-1 SIMPLIFYING RATIONAL EXPRESSIONS

Simplifying a rational expression into its lowest terms is an extremely useful skill. Its algebra is based on how we simply numerical fractions. The basic principle is developed in the first exercise.

***Exercise* #1:** Recall that to multiply fractions, one simply multiplies their numerators and denominators.

(a) Simplify the numerical fraction $\frac{18}{12}$ by first expressing it as a product of two fractions, one of which is equal to one.

(b) Simplify the algebraic fraction $\frac{x^2-9}{2x+6}$ by first expressing it as the product of two fractions (factor!), one of which is equal to one.

Every time we simplify a fraction, we are essentially finding all common factors of the numerator and denominator and dividing them to be equal to one. Key in this process is that the numerator and denominator **must be factored** and **only common factors cancel each other.** This is true whether our fraction contains monomial, binomial, or polynomial expressions.

***Exercise* #2:** Simplify each of the following monomials dividing other monomials.

(a) $\frac{3x^5y^6}{6x^8y^3}$

(b) $\frac{20x^{10}y^8}{4x^2}$

(c) $\frac{7x^3y}{21x^5y^8}$

***Exercise* #3:** Which of the following is equivalent to $\frac{10x^6y^3}{15x^2y^6}$?

(1) $\frac{2x^3}{3y^2}$

(3) $\frac{2x^4}{3y^3}$

(2) $\frac{3x^8}{2y^9}$

(4) $\frac{3x^2}{2y^3}$

When simplifying rational expressions that are more complex, always factor first, then identify common factors that can be eliminated.

***Exercise* #4:** Simplify each of the following rational expressions.

(a) $\frac{x^2+5x-14}{x^2-4}$ (b) $\frac{4x^2-1}{10x-5}$ (c) $\frac{3x^2+14x+8}{x^2-16}$

A special type of simplifying occurs whenever expressions of the form $(x-y)$ and $(y-x)$ are involved.

***Exercise* #5:** Simplify each of the following fractions.

(a) $\frac{9-6}{6-9}$ (b) $\frac{15-3}{3-15}$ (c) $\frac{a-b}{b-a}$

***Exercise* #6:** Which of the following is equivalent to $\frac{2x-10}{25-x^2}$?

(1) $\frac{-2}{x+5}$ (3) $\frac{x+5}{2}$

(2) $\frac{2-x}{5}$ (4) $\frac{2}{x-5}$

____

***Exercise* #7:** Which of the following is equivalent to $\frac{x^2-6x+9}{18-6x}$?

(1) $\frac{-x-3}{6}$ (3) $\frac{x+3}{9}$

(2) $\frac{x-3}{6}$ (4) $\frac{3-x}{6}$

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4-2 Multiplying and Dividing Rational Expressions

Multiplication of rational expressions follows the same principles as those involved in simplifying them. The process is illustrated in *Exercise* #1 with both a numerical and algebraic fraction. Notice the parallels.

***Exercise* #1:** Simplify each of the following rational expressions by factoring completely. For the numerical fraction, make sure to prime factor all numerators and denominators.

(a) $\frac{6}{8}\cdot\frac{10}{3}$

(b) $\frac{x^2-4}{x^2-x-6}\cdot\frac{3x^2+15x}{6x^2-12x}$

The ability to "cross-cancel" with fractions is a result of the two facts: (1) to multiply fractions we multiply their respective numerators and denominators and (2) multiplication is commutative. The keys to multiplication, then, are the same as that for simplifying – factor and then reduce.

***Exercise* #2:** Simplify each of the following products.

(a) $\frac{8y^7}{5x^3}\cdot\frac{10x^2}{6y^3}$

(b) $\frac{6x^2y^3}{4x^5y^2}\cdot\frac{10xy^2}{9x^5y^7}$

(c) $\frac{2x^2+12x}{4x+8}\cdot\frac{x^2-4x-12}{x^2-36}$

(d) $\frac{9-x^2}{2x^3-6x^2}\cdot\frac{4x^2-4x}{x^2+2x-3}$

Division of rational expressions continues to follow from what you have seen in previous courses. Since division by a fraction can always be thought of in terms of multiplying by it reciprocal, these problems simply involve an additional step.

***Exercise* #3:** Perform each of the following division problems. Express all answers in simplest form.

(a) $\frac{15x^2}{6y^5} \div \frac{5x^8}{2y^7}$

(b) $\frac{30y^6}{20x^3} \div \frac{24y^2}{8x}$

(c) $\frac{x^2+2x-8}{8x-16} \div \frac{x^2-16}{2x+10}$

(d) $\frac{9x^2-1}{3x^2+7x+2} \div \frac{5-15x}{x^2-5x-14}$

***Exercise* #4:** When $\frac{x^2-25}{3x}$ is divided by $\frac{x+5}{9x}$ the result is

(1) $\frac{x+5}{27x}$

(3) $\frac{x-20}{3}$

(2) $3x-15$

(4) $9x-5$

4-3 COMBINING RATIONAL EXPRESSIONS WITH ADDITION AND SUBTRACTION

Occasionally it will be important to be able to combine two or more rational expressions by addition. Keep in mind two key principles that dictate fraction addition.

TWO GUIDELINES FOR ADDITION AND SUBTRACTION OF FRACTIONS

1. Fractions must have a common denominator.
2. Denominators can only be changed by multiplying the overall fraction by one.

***Exercise* #1:** Combine each of the following fractions by first finding a common denominator. Express your answers in simplest form.

(a) $\dfrac{2x-5}{4x} + \dfrac{4x+2}{6x}$

(b) $\dfrac{4x-1}{5x} + \dfrac{x+5}{10}$

(c) $\dfrac{3}{4x} + \dfrac{1}{2x^2}$

Each of the combinations in *Exercise* #1 should have been reasonably easy because each denominator was monomial in nature. If this is not the case, then it is wise to **factor** the denominators before trying to find a common denominator.

***Exercise* #2:** Combine each of the following fractions by factoring the denominators first. Then find a common denominator and add.

(a) $\dfrac{4}{5y-15} + \dfrac{5}{y^2-9}$

(b) $\dfrac{x-3}{x^2-9x+20} + \dfrac{2}{x^2-6x+8}$

Subtraction of rational expressions is especially challenging because of errors that naturally arise when students forget to distribute the subtraction (or the multiplication by -1). Still, with careful execution, these problems are no different than their addition counterparts.

***Exercise* #3:** Perform each of the following subtraction problems. Express your answers in simplest form.

(a) $\frac{3x+7}{x^2-4}-\frac{x+3}{x^2-4}$

(b) $\frac{x-3}{4x^2-1}-\frac{2}{10x+5}$

(c) $\frac{x}{x^2-4}-\frac{6}{x^2+8x-20}$

(d) $\frac{x-2}{x^2+5x+4}-\frac{8}{x^2+12x+32}$

***Exercise* #4:** Which of the following is equivalent to $\frac{1}{x-1}-\frac{1}{x}$?

(1) $\frac{x}{x-1}$

(3) $\frac{1}{x^2-x}$

(2) $\frac{1}{x-x^2}$

(4) $\frac{x}{x^2-1}$

4-4 Complex Fractions

Complex fractions are simply defined as fractions that have other fractions within their numerators and/or denominators. To simplify these fractions means to remove these minor fractions and then eliminate any common factors. The key, as always, is to multiply by the number one in ways that simplify the fraction.

***Exercise* #1:** Consider the complex fraction $\dfrac{\frac{1}{9}+\frac{1}{18}}{\frac{1}{3}}$.

(a) What is the least common denominator amongst the three minor fractions?

(b) Multiply the numerator and denominator of the major fraction by your answer in part (a) and then simplify your result.

(c) Why is it acceptable to perform the operation in part (b)? What number are you effectively multiplying by?

By multiplying the major fraction by the number one, by using the least common denominator, we will always eliminate the minor fractions (by turning them into integer expressions).

***Exercise* #2:** Simplify each of the following complex fractions.

(a) $\dfrac{\frac{1}{2}-\frac{1}{10}}{\frac{2}{5}}$

(b) $\dfrac{\frac{2}{3}+\frac{2}{x}}{\frac{5}{3}+\frac{5}{x}}$

(c) $\dfrac{\frac{3}{8}+\frac{1}{4x}}{\frac{7}{2x}+\frac{3}{4}}$

These types of problems can certainly involve more complicated secondary simplification. Don't forget the primary use of factoring in order to simplify.

***Exercise* #3:** Simplify each of the following complex fractions.

(a) $\dfrac{\frac{1}{2}-\frac{2}{x^2}}{\frac{3}{2x}-\frac{3}{x^2}}$ (b) $\dfrac{\frac{2}{5}-\frac{2}{x}}{\frac{1}{5x}-\frac{1}{x^2}}$ (c) $\dfrac{\frac{x}{12}+\frac{1}{6}-\frac{2}{x}}{\frac{x}{12}-\frac{4}{3x}}$

If the denominators of the minor fractions become more complex, be sure to factor them first, just as you did with the addition and subtraction in the previous lesson.

***Exercise* #4:** Simplify each of the following complex fractions.

(a) $\dfrac{\frac{4}{x+2}+\frac{2}{x-4}}{\frac{12x-24}{x^2-2x-8}}$ (b) $\dfrac{\frac{x}{x+6}-\frac{1}{x+2}}{\frac{x^2-4}{x^2+8x+12}}$

4-5 SOLVING FRACTIONAL EQUATIONS

Equations involving fractions or rational expressions arise frequently in mathematics. The key to working with them is to manipulate the equation, typically by multiplying both sides of it by some quantity, that eliminates the fractional nature of the equation. The most common form of this practice is "cross-multiplying."

***Exercise* #1:** Use the technique of cross multiplication to solve each of the following equations.

(a) $\frac{4x+5}{2}=\frac{x-1}{5}$

(b) $\frac{x+1}{2-x}=\frac{2}{x-6}$

Since this technique should be familiar to students at this point, we will move onto a less familiar method when more than two fractions are involved. The next exercise will illustrate the process.

***Exercise* #2:** Consider the equation $\frac{1}{2}-\frac{9}{4x}=\frac{3}{4x}$.

(a) What is the least common denominator for all three fractions in this equations?

(b) Multiply both sides of this equation by the LCD to "clear" the equation of the denominators. Now, solve the resulting linear equation.

It is very important to note the similarities and differences between this technique and the one employed to simplify complex fractions. With complex fractions we multiplied by one in creative ways. Here we are multiplying both sides of an equation by a quantity that removes the fractional nature of the equation.

***Exercise* #3:** Which of the following values of x solves: $\frac{x-4}{6}+\frac{x-2}{10}=\frac{31}{15}$?

(1) $x=14$

(2) $x=6$

(3) $x=-8$

(4) $x=11$

______

These equations can involve quadratic as well as root expressions. The key, though, remains the same – multiplying both sides of the equation by the same quantity.

***Exercise* #4:** Solve each of the following equations for all values of *x*.

(a) $\frac{1}{10}-\frac{1}{x}=\frac{1}{5x}-\frac{2}{x^2}$

(b) $\frac{1}{2}+\frac{3}{x}-\frac{1}{x^2}=\frac{1}{4x}+\frac{1}{2x^2}$

Because fractional equations often involve denominators containing variables, it is important that we check to see if any solutions to the equation make it undefined. These represent further examples of **extraneous roots**.

***Exercise* #5:** Solve and reject any extraneous roots.

(a) $\frac{x+1}{x+5}+\frac{18}{x^2+8x+15}=\frac{9}{x+3}$

(b) $\frac{4}{x^2+4x-12}+\frac{x-1}{x+6}=\frac{1}{x-2}$

Work Problems: $\frac{Together}{Alone} + \frac{Together}{Alone} = 1$ *job completed*

RULES FOR SOLVING WORK PROBLEMS:

1. **To solve work problems, you need to work with the same unit of measure within each problem. For example, you cannot mix hours and minutes in the same equation.**
2. **You need to find the fractional part of the job that would be done in one unit of time, such as 1 minute or 1 hour. If a person can do a complete job in 3 days, he can do 1/3 of it in 1 day.**

Example: Mark can dig a ditch in 4 hours. Greg can dig the same ditch in 3 hours. How long would it take them to dig it together?

1. Write an equation: $\frac{Together}{Alone} + \frac{Together}{Alone} = 1$ OR $\frac{T}{A} + \frac{T}{A} = 1$

2. Substitute your values in the equation:

3. Solve!

Try These!!!

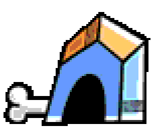
1. Jaron can paint a fence in 3 hours and his friend Tyrek can paint it in 5 hours. How long would it take them to do the job if they work together?

2. It takes Ashly 3 days to cultivate the garden. It takes Ivanna twice as long to do the same job. How long does it take them to do the job if they work together?

3. Heckel can plow a field in 6 hours. If his brother Jeckel helps him, it will take 4 hours. How long would it take Jeckel to do the job alone?

4. Jordan can paint a fence in 8 hours. Kejuan can paint the same fence in 4 hours. How long will it take them to do the job if they work together?

5. If Dal can build a doghouse in 5 hours, and together he and Yves can build it in 2 hours, how many hours would it take Yves alone to build the same doghouse?



6. One machine can complete a job in 10 minutes. If this machine and an older machine do the same job together, the job can be completed in 6 minutes. How long would it take the older machine to do the job?

7. Working alone, Janell can pick 40 bushels of apples in 11 hours. Shanice can pick the same amount in 14 hours. How long would it take them if they worked together?

4-6 Graphing Rational Functions

Rational Equations:

Sketch some of them!

<u>Find the Domain</u>

Removable (Holes)

Vertical Asymptote(s)

$y = \frac{x-4}{2x^2-8x}$	$y = \frac{6x^2+7x-3}{x^2+4}$	$y = \frac{x^2+5}{x+1}$

<u>Find the x- and y-intercept(s)</u>

Y-intercept (x = 0)

X-intercept (y = 0)

$y = \frac{x-4}{2x^2-8x}$	$y = \frac{6x^2+7x-3}{x^2+4}$	$y = \frac{x^2+5}{x+1}$

<u>Horizontal Asymptotes</u>

Degree in the numerator is bigger:

Degree in the denominator is bigger:

Degrees are the same:

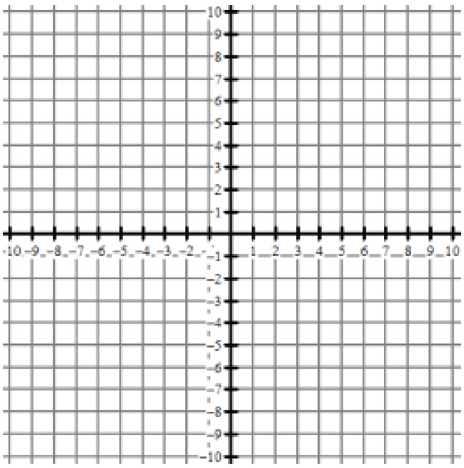
$y = \frac{x-4}{2x^2-8x}$	$y = \frac{6x^2+7x-3}{x^2+4}$	$y = \frac{x^2+5}{x+1}$

<u>Slant (Oblique) Asymptotes</u>

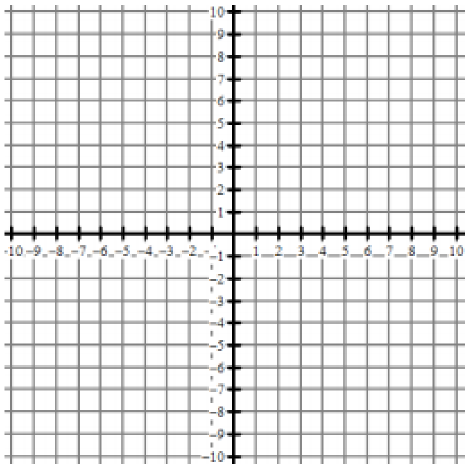
$$y = \frac{x^2 + 5}{x + 1}$$

Put it all together

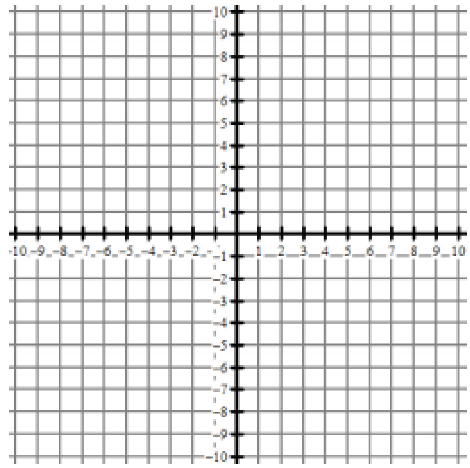
$y = \frac{x-4}{2x^2-8x}$



$y = \frac{6x^2+7x-3}{x^2+4}$



$y = \frac{x^2+5}{x+1}$

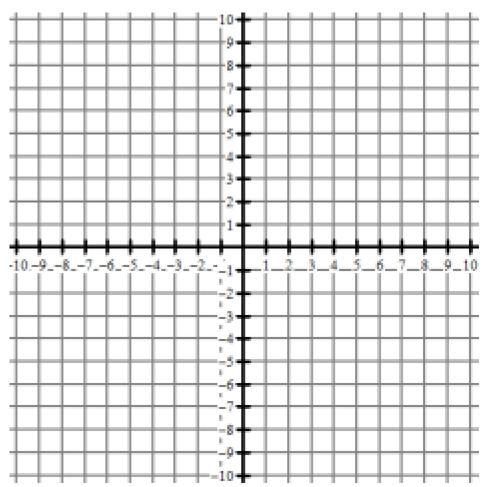


$$y = \frac{2x^3-x^2-x}{x^2+2x+1}$$

Hole/Vertical Asymptotes:

Y-Int: X-int:

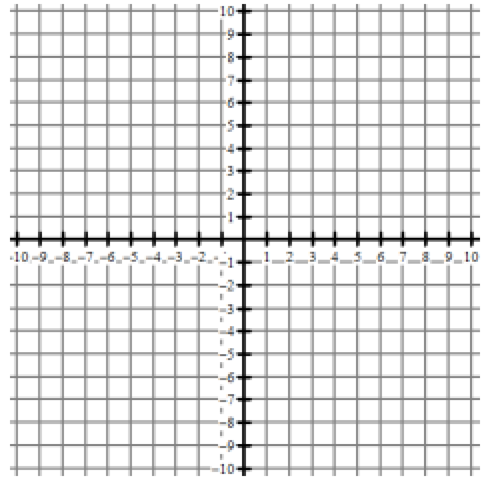
Horizontal/Slant Asymptote:



Rational Function	Points of Discontinuity	Holes	Vertical Asymptotes	Horizontal Asymptotes	Sketch of the Graph
$y=\frac{x+3}{(x-2)(x+3)}$	x = ________				
$y = \frac{2x^2+4x}{x+2}$	x = ________				
$y = \frac{2x}{x+5}$	x = ________				
$y = \frac{x}{x-3}$	x = ________				
$y = \frac{x^2-3x-28}{x+4}$	x = ________				
$y = \frac{(x-2)(x+1)}{x(x-5)(x+1)}$	x = ________				

You try!

$$y = \frac{x^2 - 4x - 12}{x^2 + 5x + 6}$$



Hole/Vertical Asymptotes:

Y-Int: X-int:

Horizontal/Slant Asymptote:

PRACTICE

Directions: Find any holes or vertical asymptotes.		
1) $y = \frac{1}{x^2-6x-16}$	2) $y = \frac{2x^2+11x-6}{x^2+2x-24}$	3) $y = \frac{2x^2-6x}{9x-3x^2}$

Directions: Find the x- and y-intercept(s)		
4) $y = \frac{2x-3}{4x+5}$	5) $y = \frac{6x^2+x-12}{x^2-13x-40}$	6) $y = \frac{x^2+x-30}{x^2-8x+15}$

Directions: Find any horizontal asymptotes.		
7) $y = \frac{4x^3+7x-12}{2x-7}$	8) $y = \frac{8x-3}{2x+9}$	9) $y = \frac{3x^2-4x+9}{4x^3+8x^2-10x+1}$

Directions: Find the slant asymptote (if it exists).

10) $y = \frac{6x^3+8x^2-7x}{2x^2-3x+1}$	11) $y = \frac{2x^2+11x-6}{x^2+2x-24}$	12) $y = \frac{x^2+6x-10}{2x-4}$

Directions: Find the information need and sketch. Include all relevant information on your graph.

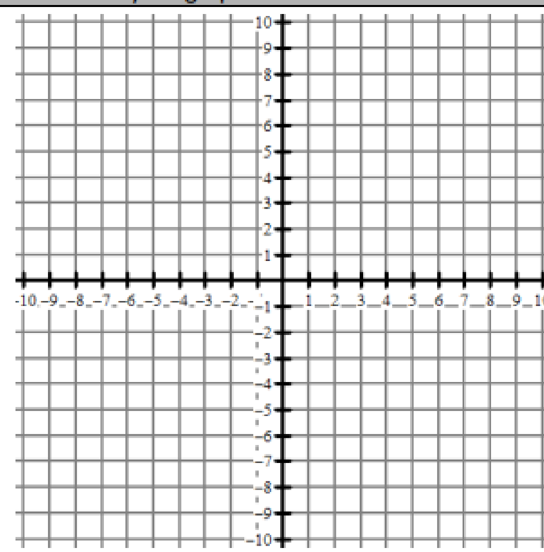
13) $y = \frac{9}{x^2+1}$

Hole/Vertical Asymptotes:

Y-Int:

X-int:

Horizontal/Slant Asymptote:



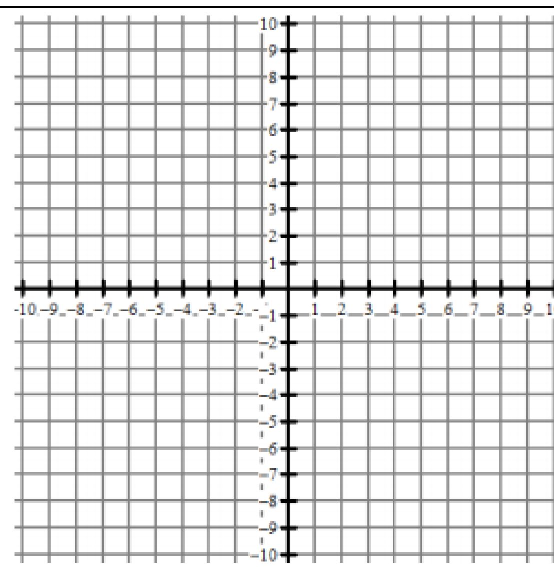
14) $y = \frac{x^3-x^2-20x}{x^2-2x-3}$

Hole/Vertical Asymptotes:

Y-Int:

X-int:

Horizontal/Slant Asymptote:

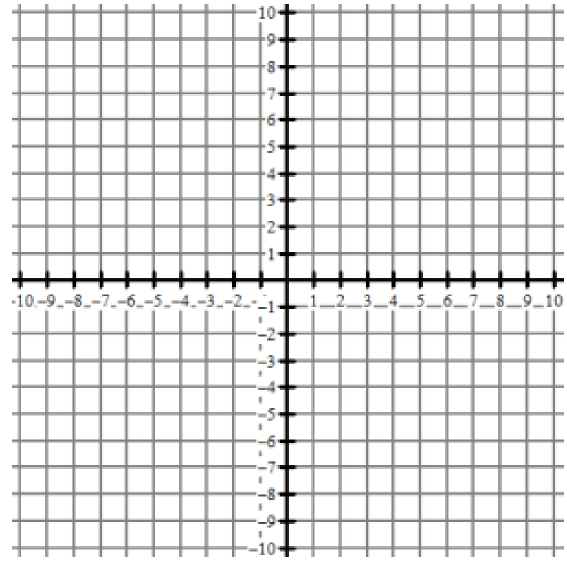


15) $y = \frac{2x+8}{x^2-2x-24}$

Hole/Vertical Asymptotes:

Y-Int: X-int:

Horizontal/Slant Asymptote:



16) $y = \frac{x-2}{x^2-2x-3}$

Hole/Vertical Asymptotes:

Y-Int: X-int:

Horizontal/Slant Asymptote: