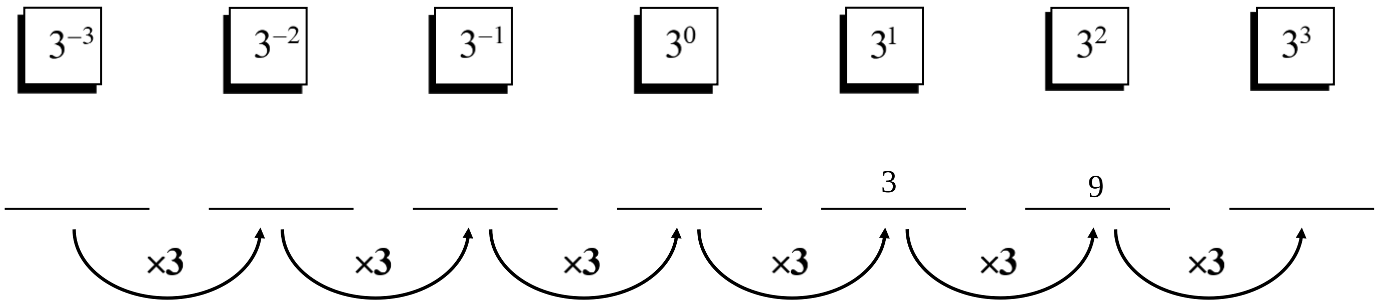
Name: ______________________________ Date: _______________

5-1 Integer Exponents
Math III

We just finished our review of linear functions. Linear functions are those that grow by equal differences for equal intervals. In this unit we will concentrate on exponential functions which grow by equal factors for equal intervals. To understand exponential functions, we first need to understand exponents.

***Exercise* #1:** The following sequence shows powers of 3 by repeatedly multiplying by 3. Fill in the missing blanks.



This pattern can be duplicated for any **base** raised to any **integer exponent**. Because of this we can now define positive, negative, and zero exponents in terms of multiplying the number 1 repeatedly or dividing the number 1 repeatedly.

Integer Exponent Definitions

If n is any positive integer then:

1. $b^n = 1 \cdot \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b \cdot b}_{n\text{-times}}$

2. $b^0 = 1$

3. $b^{-n} = \dfrac{1}{\underbrace{b \cdot b \cdot b \cdot \ldots \cdot b \cdot b}_{n\text{-times}}} = \dfrac{1}{b^n}$

***Exercise* #2:** Given the exponential function $f(x) = 20(2)^x$ evaluate each of the following without using your calculator. Show the calculations that lead to your final answer.

(a) $f(2)$

(b) $f(0)$

(c) $f(-2)$

(d) When x increases by 3, by what factor does y increase? Explain your answer.

There are many basic **exponent properties or laws** that are critically important and that can be investigated using integer exponent examples. Two of the very important ones we will see next.

***Exercise* #3:** For each of the following, write the product as a single exponential expression. Write (a) and (b) as extended products first (if necessary).

(a) $2^3 \cdot 2^4$

(b) $2^6 \cdot 2^2$

(c) $2^m \cdot 2^n$

It's clear why the exponent law that you generalized in part (c) works for positive integer exponents. But, does it also make sense within the context of our negative exponents?

***Exercise* #4:** Consider now the product $2^3 \cdot 2^{-1}$.

(a) Use the exponent law found in Exercise 3(c) to write this as a single exponential expression.

(b) Evaluate $2^3 \cdot 2^{-1}$ by first rewriting 2^3 and 2^{-1} and then simplifying.

(c) Do your answers from (a) and (b) support the extension of the **Addition Property of Exponents** to negative powers as well? Explain.

Let's look at another important exponent property.

***Exercise* #5:** For each of the following, write the exponential expression in the form 3^x. Write (a) and (b) as extended products first (if necessary).

(a) $\left(3^2\right)^3$

(b) $\left(3^4\right)^2$

(c) $\left(3^m\right)^n$

Again, let's look at how the **Product Property of Exponents** still holds for negative exponents.

***Exercise* #6:** Consider the expression $\left(3^{-2}\right)^4$. Show this expression is equivalent to 3^{-8} by first rewriting 3^{-2} in fraction form.

Name: ____________________________________ Date: __________________

RATIONAL EXPONENTS
MATH III

When you first learned about exponents, they were always **positive integers**, and just represented **repeated multiplication**. And then we had to go and introduce **negative exponents**, which really just represent **repeated division**. Today we will introduce **rational (or fractional) exponents** and extend your exponential knowledge that much further.

***Exercise* #1:** Recall the **Product Property of Exponents** and use it to rewrite each of the following as a simplified exponential expression. There is no need to find a final numerical value.

(a) $\left(2^3\right)^4$ (b) $\left(5^{-2}\right)^5$ (c) $\left(3^7\right)^0$ (d) $\left(\left(4^2\right)^{-2}\right)^2$

We will now use the Product Property to extend our understanding of exponents to include **unit fraction** exponents (those of the form $1/n$ where n is a positive integer).

***Exercise* #2:** Consider the expression $16^{1/2}$.

(a) Apply the Product Property to simplify $\left(16^{1/2}\right)^2$. What other number squared yields 16?

(b) You can now say that $16^{1/2}$ is equivalent to what more familiar quantity?

This is remarkable! An exponent of $1/2$ is equivalent to a square root of a number!!!

***Exercise* #3:** Test the equivalence of the $1/2$ exponent to the square root by using your calculator to evaluate each of the following. Be careful in how you enter each expression.

(a) $25^{1/2} =$ (b) $81^{1/2} =$ (c) $100^{1/2} =$

We can extend this now to all levels of roots, that is square roots, cubic roots, fourth roots, etcetera.

UNIT FRACTION EXPONENTS

For n given as a positive integer: $b^{1/n} = \sqrt[n]{b}$

***Exercise* #4:** Rewrite each of the following using roots instead of fractional exponents. Then, if necessary, evaluate using your calculator to guess and check to find the roots (don't use the generic root function). Check with your calculator.

(a) $125^{1/3}$ (b) $16^{1/4}$ (c) $9^{-1/2}$ (d) $32^{-1/5}$

We can now combine traditional integer powers with unit fractions in order interpret any exponent that is a **rational number**, i.e. the **ratio of two integers**. The next exercise will illustrate the thinking. Remember, we want our exponent properties to be consistent with the structure of the expression.

***Exercise* #5:** Let's think about the expression $4^{3/2}$.

(a) Fill in the missing blank and then evaluate this expression:

$$4^{3/2} = (\quad)^{1/2}$$

(b) Fill in the missing blank and then evaluate this expression:

$$4^{3/2} = (\quad)^{3}$$

(c) Verify both (a) and (b) using your calculator.

(d) Evaluate $27^{2/3}$ without your calculator. Show your thinking. Verify with your calculator.

RATIONAL EXPONENT CONNECTION TO ROOTS

For the rational number m/n we define $b^{m/n}$ to be: $\sqrt[n]{b^m}$ or $\left(\sqrt[n]{b}\right)^m$.

***Exercise* #6:** Evaluate each of the following exponential expressions involving rational exponents without the use of your calculator. Show your work. Then, check your final answers with the calculator.

(a) $16^{3/4}$ (b) $25^{3/2}$ (c) $8^{-2/3}$

5- 2 EXPONENTIAL FUNCTION BASICS
MATH III

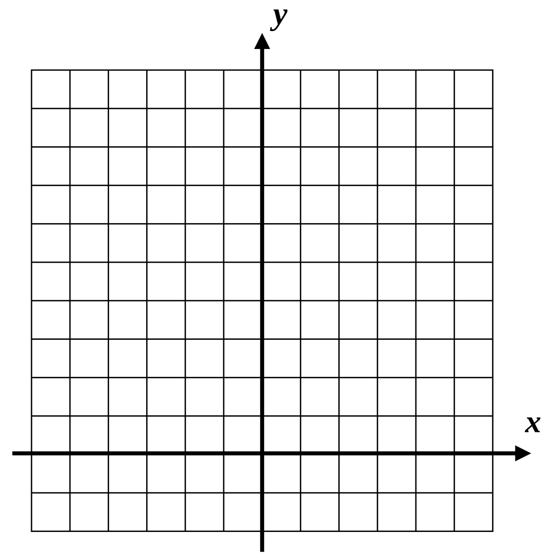
You studied exponential functions extensively in Common Core Algebra I. Today's lesson will review many of the basic components of their graphs and behavior. Exponential functions, those whose exponents are variable, are extremely important in mathematics, science, and engineering.

BASIC EXPONENTIAL FUNCTIONS

$y = b^x$ where $b > 0$ and $b \neq 1$

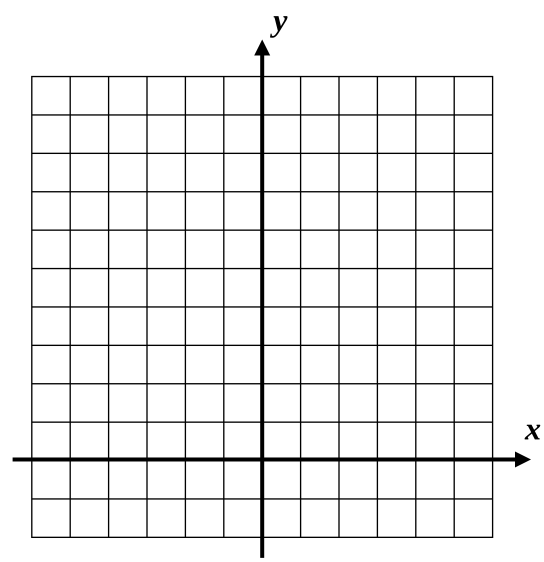
***Exercise* #1:** Consider the function $y = 2^x$. Fill in the table below without using your calculator and then sketch the graph on the grid provided.

x	$y = 2^x$
-3	
-2	
-1	
0	
1	
2	
3	



***Exercise* #2:** Now consider the function $y = \left(\frac{1}{2}\right)^x$. Using your calculator to help you, fill out the table below and sketch the graph on the axes provided.

x	$y = \left(\frac{1}{2}\right)^x$
-3	
-2	
-1	
0	
1	
2	
3	

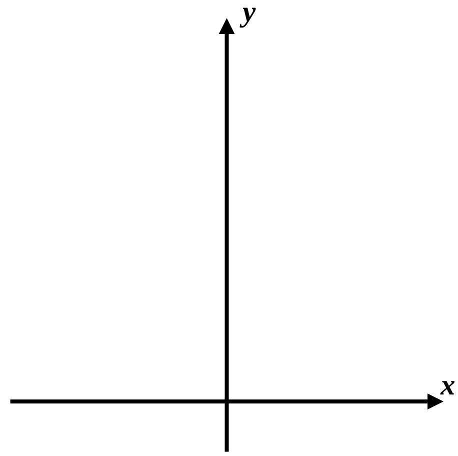


***Exercise* #3:** Based on the graphs and behavior you saw in *Exercises* #1 and #2, state the domain and range for an exponential function of the form $y = b^x$.

Domain (input set): Range (output set):

***Exercise* #4:** Are exponential functions one-to-one? How can you tell? What does this tell you about their inverses?

***Exercise* #5:** Now consider the function $y = 7(3)^x$.

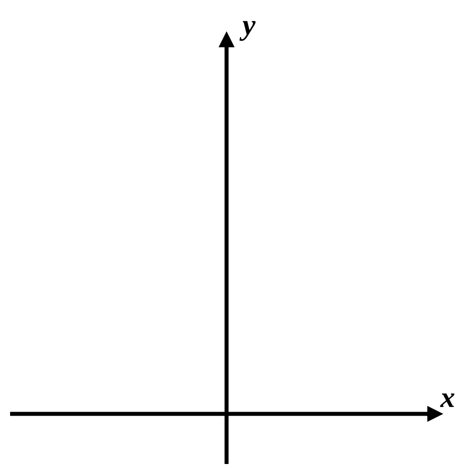


(a) Determine the y-intercept of this function algebraically. Justify your answer.

(b) Does the exponential function increase or decrease? Explain your choice.

(c) Create a rough sketch of this function, labeling its y-intercept.

***Exercise* #6:** Consider the function $y = \left(\frac{1}{3}\right)^x + 4$.



(a) How does this function's graph compare to that of $y = \left(\frac{1}{3}\right)^x$? What does adding 4 do to a function's graph?

(b) Determine this graph's y-intercept algebraically. Justify your answer.

(c) Create a rough sketch of this function, labeling its y-intercept.

5-3 THE METHOD OF COMMON BASES
MATH III

There are very few algebraic techniques that **do not involve technology** to solve equations that contain **exponential expressions**. In this lesson we will look at one of the few, known as **The Method of Common Bases**.

***Exercise* #1:** Solve each of the following simple exponential equations by writing each side of the equation using **a common base**.

(a) $2^x = 16$ (b) $3^x = 27$ (c) $5^x = \frac{1}{25}$ (d) $16^x = 4$

In each of these cases, even the last, more challenging one, we could manipulate the right-hand side of the equation so that it shared a **common base** with the left-hand side of the equation. We can exploit this fact by manipulating both sides so that they have a common base. First, though, we need to review an exponent law.

***Exercise* #2:** Simplify each of the following exponential expressions.

(a) $\left(2^3\right)^x$ (b) $\left(3^2\right)^{4x}$ (c) $\left(5^{-1}\right)^{3x-7}$ (d) $\left(4^{-3}\right)^{1-x^2}$

***Exercise* #3:** Solve each of the following equations by finding a common base for each side.

(a) $8^x = 32$ (b) $9^{2x+1} = 27$ (c) $125^x = \left(\frac{1}{25}\right)^{4-x}$

***Exercise* #4:** Which of the following represents the solution set to the equation $2^{x^2-3} = 64$?

(1) $\{\pm 3\}$ (3) $\{\pm\sqrt{11}\}$

(2) $\{0, 3\}$ (4) $\{\pm\sqrt{35}\}$

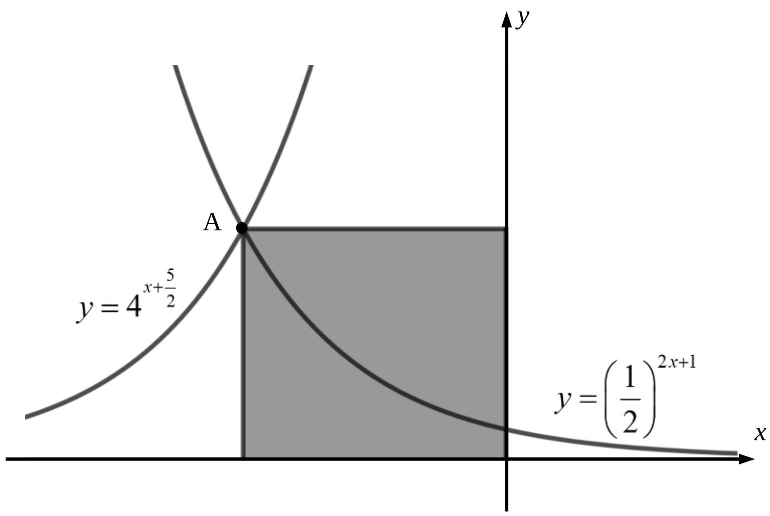
______

This technique can be used in any situation where all bases involved can be written with a common base. In a practical sense, this is rather rare. Yet, these types of algebraic manipulations help us see the **structure** in **exponential expressions**. Try to tackle the next, more challenging, problem.

***Exercise* #5:** Two exponential curves, $y=4^{x+\frac{5}{2}}$ and $y=\left(\frac{1}{2}\right)^{2x+1}$ are shown below. They intersect at point A. A rectangle has one vertex at the origin and the other at A as shown. We want to find its area.

(a) Fundamentally, what do we need to know about a rectangle to find its area?

(b) How would knowing the coordinates of point A help us find the area?



(c) Find the area of the rectangle algebraically using the Method of Common Bases. Show your work carefully.

***Exercise* #6:** At what x coordinate will the graph of $y=25^{x-a}$ intersect the graph of $y=\left(\frac{1}{125}\right)^{3x+1}$? Show the work that leads to your choice.

(1) $x=\frac{5a-1}{3}$ (3) $x=\frac{-2a+1}{5}$

(2) $x=\frac{2a-3}{11}$ (4) $x=\frac{5a+3}{2}$

____

5-4 EXPONENTIAL MODELING WITH PERCENT GROWTH AND DECAY
MATH III

Exponential functions are very important in modeling a variety of real world phenomena because certain things either increase or decrease by **fixed percentages** over given units of time. You looked at this in Common Core Algebra I and in this lesson we will review much of what you saw.

***Exercise* #1:** Suppose that you deposit money into a savings account that receives 5% interest per year on the amount of money that is in the account for that year. Assume that you deposit $400 into the account initially.

(a) How much will the savings account increase by over the course of the year?

(b) How much money is in the account at the end of the year?

(c) By what single number could you have multiplied the $400 by in order to calculate your answer in part (b)?

(d) Using your answer from part (c), determine the amount of money in the account after 2 and 10 years. Round all answers to the nearest cent when needed.

(e) Give an equation for the amount in the savings account $S(t)$ as a function of the number of years since the $400 was invested.

(f) Using a table on your calculator determine, to the nearest year, how long it will take for the initial investment of $400 to double. Provide evidence to support your answer.

The thinking process from *Exercise* #1 can be generalized to any situation where a quantity is increased by a fixed percentage over a fixed interval of time. This pattern is summarized below:

INCREASING EXPONENTIAL MODELS

If quantity *Q* is known to increase by a fixed percentage *p*, in decimal form, then *Q* can be modeled by

$$Q(t)=Q_0(1+p)^t$$

where Q_0 represents the amount of *Q* present at $t=0$ and *t* represents time.

***Exercise* #2:** Which of the following gives the savings *S* in an account if $250 was invested at an interest rate of 3% per year?

(1) $S=250(4)^t$ (3) $S=(1.03)^t+250$

(2) $S=250(1.03)^t$ (4) $S=250(1.3)^t$

____

Decreasing exponentials are developed in the same way, but have the percent subtracted, rather than added, to the base of 100%. Just remember, you are ultimately multiplying by the percent of the original that you will have after the time period elapses.

***Exercise* #3:** State the multiplier (base) you would need to multiply by in order to decrease a quantity by the given percent listed.

(a) 10%   (b) 2%   (c) 25%   (d) 0.5%

DECREASING EXPONENTIAL MODELS

If quantity *Q* is known to decrease by a fixed percentage *p*, in decimal form, then *Q* can be modeled by

$$Q(t)=Q_0(1-p)^t$$

where Q_0 represents the amount of *Q* present at $t=0$ and *t* represents time.

***Exercise* #4:** If the population of a town is decreasing by 4% per year and started with 12,500 residents, which of the following is its projected population in 10 years? Show the exponential model you use to solve this problem.

(1) 9,230   (3) 18,503

(2) 76   (4) 8,310

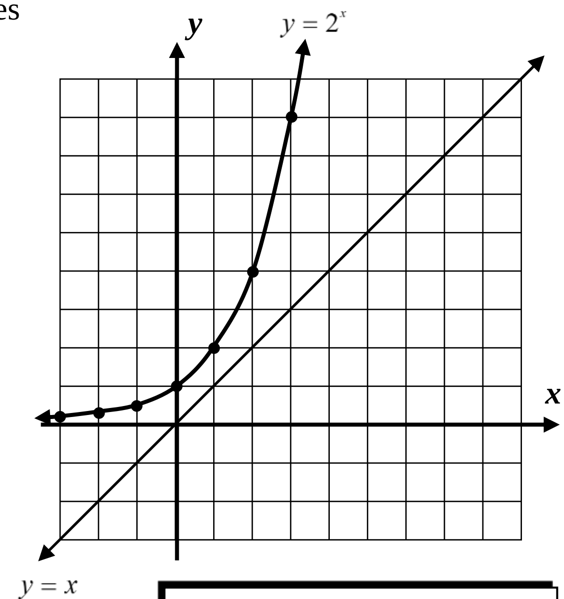
____

***Exercise* #5:** The stock price of WindpowerInc is increasing at a rate of 4% per week. Its initial value was $20 per share. On the other hand, the stock price in GerbilEnergy is crashing (losing value) at a rate of 11% per week. If its price was $120 per share when Windpower was at $20, after how many weeks will the stock prices be the same? Model both stock prices using exponential functions. Then, find when the stock prices will be equal graphically. Draw a well labeled graph to justify your solution.

5-5 INTRODUCTION TO LOGARITHMS
MATH III

Exponential functions are of such importance to mathematics that their inverses, functions that "reverse" their action, are important themselves. These functions, known as **logarithms**, will be introduced in this lesson.

***Exercise* #1:** The function $f(x)=2^x$ is shown graphed on the axes below along with its table of values.



x	–3	–2	–1	0	1	2	3
$f(x)=2^x$	$1/8$	$1/4$	$1/2$	1	2	4	8

(a) Is this function one-to-one? Explain your answer.

(b) Based on your answer from part (a), what must be true about the inverse of this function?

Notice that, as always, the graphs of $f(x)$ and $f^{-1}(x)$ are symmetric across $y=x$

(c) Create a table of values below for the inverse of $f(x)=2^x$ and plot this graph on the axes given.

x							
$f^{-1}(x)$							

(d) What would be the first step to find an equation for this inverse algebraically? Write this step down and then stop.

<u>Defining Logarithmic Functions</u> – The function $y=\log_b x$ is the name we give the inverse of $y=b^x$. For example, $y=\log_2 x$ is the inverse of $y=2^x$. Based on *Exercise* #1(d), we can write an **equivalent exponential equation** for each logarithm as follows:

$$y=\log_b x \text{ is the same as } b^y=x$$

Based on this, we see that a logarithm gives as its output (y-value) the exponent we must raise b to in order to produce its input (x-value).

***Exercise* #2:** Evaluate the following logarithms. If needed, write an equivalent exponential equation. Do as many as possible without the use of your calculator.

(a) $\log_2 8$

(b) $\log_4 16$

(c) $\log_5 625$

(d) $\log_{10} 100{,}000$

(e) $\log_6 \left(\frac{1}{36}\right)$

(f) $\log_2 \left(\frac{1}{16}\right)$

(g) $\log_5 \sqrt{5}$

(h) $\log_3 \sqrt[5]{9}$

It is critically important to understand that logarithms **give exponents as their outputs**. We will be working for multiple lessons on logarithms and a basic understanding of their inputs and outputs is critical.

***Exercise* #3:** If the function $y = \log_2(x+8)+9$ was graphed in the coordinate plane, which of the following would represent its *y*-intercept?

(1) 12

(2) 13

(3) 8

(4) 9

____

***Exercise* #4:** Between which two consecutive integers must $\log_3 40$ lie?

(1) 1 and 2

(2) 2 and 3

(3) 3 and 4

(4) 4 and 5

____

<u>Calculator Use and Logarithms</u> – Most calculators only have two logarithms that they can evaluate directly. One of them, $\log_{10} x$, is so common that it is actually called the **common log** and typically is written without the base 10.

$$\log x = \log_{10} x \quad \text{(The Common Log)}$$

***Exercise* #5:** Evaluate each of the following using your calculator.

(a) $\log 100$

(b) $\log\left(\frac{1}{1000}\right)$

(c) $\log \sqrt{10}$

5-6 GRAPHS OF LOGARITHMS
MATH III

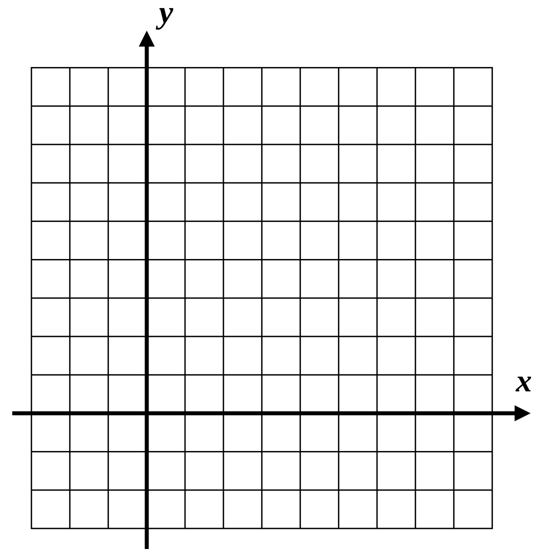
The vast majority of logarithms that are used in the real world have bases greater than one; the pH scale that we saw on the last homework assignment is a good example. In this lesson we will further explore graphs of these logarithms, including their construction, transformations, and domains and ranges.

***Exercise* #1:** Consider the logarithmic function $y = \log_3 x$ and its inverse $y = 3^x$.

(a) Construct a table of values for $y = 3^x$ and then use this to construct a table of values for the function $y = \log_3 x$.

x	-2	-1	0	1	2
$y = 3^x$					

x					
$y = \log_3 x$					



(b) Graph $y = 3^x$ and $y = \log_3 x$ on the grid given. Label with equations.

(c) State the natural domain and range of $y = 3^x$ and $y = \log_3 x$.

$y = 3^x$

Domain:

Range:

$y = \log_3 x$

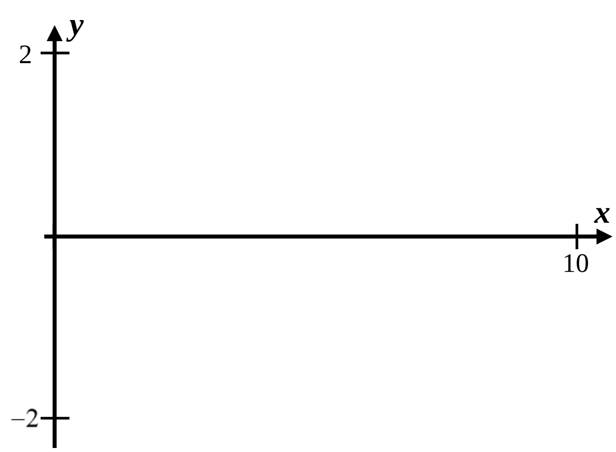
Domain:

Range:

***Exercise* #2:** Using your calculator, sketch the graph of $y = \log_{10} x$ on the axes below. Label the x-intercept. State the domain and range of $y = \log_{10} x$.

Domain:

Range:



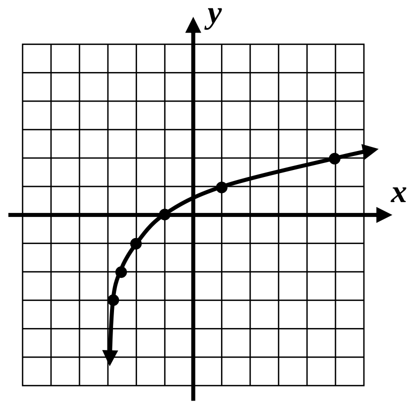
***Exercise* #3:** Which of the following equations describes the graph shown below? Show or explain how you made your choice.

(1) $y = \log_3(x+2) - 1$

(2) $y = \log_2(x-3) + 1$

(3) $y = \log_2(x+3) - 1$

(4) $y = \log_3(x+3) - 1$



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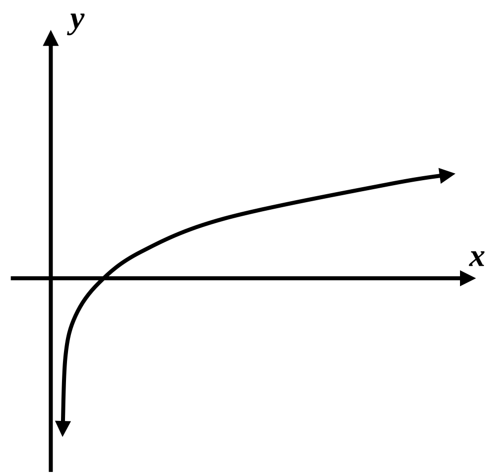
The fact that finding the logarithm of a non-positive number (negative or zero) is not possible in the real number system allows us to find the domains of a variety of logarithmic functions.

***Exercise* #4:** Determine the domain of the function $y = \log_2(3x-4)$. State your answer in set-builder notation.

All logarithms with bases larger than 1 are **always increasing**. This increasing nature can be seen by calculating their **average rate of change**.

***Exercise* #5:** Consider the common log, or log base 10, $f(x) = \log(x)$.



(a) Set up and evaluate an expression for the average rate of change of $f(x)$ over the interval $1 \le x \le 10$

(b) Set up and evaluate an expression for the average rate of change of $f(x)$ over the interval $1 \le x \le 100$.

(c) What do these two answers tell you about the changing slope of this function?

5-7 Logarithm Laws
Math III

Logarithms have properties, just as exponents do, that are important to learn because they allow us to solve a variety of problems where logarithms are involved. Keep in mind that since logarithms give exponents, the laws that govern them should be similar to those that govern exponents. Below is a summary of these laws.

Exponent and Logarithm Laws

Law	Exponent Version	Logarithm Version
Product	$b^x \cdot b^y = b^{x+y}$	$\log_b(x \cdot y) = \log_b x + \log_b y$
Quotient	$\dfrac{b^x}{b^y} = b^{x-y}$	$\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
Power	$\left(b^x\right)^y = b^{x \cdot y}$	$\log_b\left(x^y\right) = y \cdot \log_b x$

***Exercise* #1:** Which of the following is equal to $\log_3(9x)$?

(1) $\log_3 2 + \log_3 x$ (3) $2 + \log_3 x$

(2) $2\log_3 x$ (4) $x + \log_3 2$

***Exercise* #2:** The expression $\log\left(x^2/1000\right)$ can be written in equivalent form as

(1) $2\log x - 3$ (3) $2\log x - 6$

(2) $\log 2x - 3$ (4) $\log 2x - 6$

***Exercise* #3:** If $a = \log 3$ and $b = \log 2$ then which of the following correctly expresses the value of $\log 12$ in terms of a and b?

(1) $a^2 + b$ (3) $2a + b$

(2) $a + b^2$ (4) $a + 2b$

***Exercise* #4:** Which of the following is equivalent to $\log_2\left(\sqrt{x}/y^5\right)$?

(1) $\sqrt{\log_2 x} - 5\log_2 y$ (3) $\frac{1}{2}\log_2 x - 5\log_2 y$

(2) $2\log_2 x + 5\log_2 y$ (4) $2\log_2 x - 5\log_2 y$

***Exercise* #5:** The value of $\log_3\left(\sqrt{5}/27\right)$ is equal to

(1) $\dfrac{\log_3 5-6}{2}$ (3) $\dfrac{\log_3 5-3}{2}$

(2) $2\log_3 5+3$ (4) $2\log_3 5-3$ ________

***Exercise* #6:** If $f(x)=\log(x)$ and $g(x)=100x^3$ then $f(g(x))=$

(1) $100\log x$ (3) $300\log x$

(2) $6+\log x$ (4) $2+3\log x$

____

***Exercise* #7:** The logarithmic expression $\log_2\sqrt{32x^7}$ can be rewritten as

(1) $\sqrt{\log_2 35x}$ (3) $\sqrt{5+7\log_2 x}$

(2) $\dfrac{5+7\log_2 x}{2}$ (4) $\dfrac{35+\log_2 x}{2}$

____

***Exercise* #8:** If $\log 7=k$ then $\log(4900)$ can be written in terms of k as

(1) $2(k+1)$ (3) $2(k-3)$

(2) $2k-1$ (4) $2k+1$ ________

The logarithm laws are important for future study in mathematics and science. Being fluent with them is essential. Arguably, the most important of the three laws is the power law. In the next exercise, we will examine it more closely.

***Exercise* #9:** Consider the expression $\log_2\left(8^x\right)$.

(a) Using the third logarithm law (the Product Law), rewrite this as equivalent product and simplify.

(b) Test the equivalency of these two expressions for $x=0,1,$ and 2.

(c) Show that $\log_2\left(8^x\right)=3x$ by rewriting 8 as 2^3.

5-8 SOLVING EXPONENTIAL EQUATIONS USING LOGARITHMS
MATH III

Earlier in this unit, we used the **Method of Common Bases** to solve exponential equations. This technique is quite limited, however, because it requires the two sides of the equation to be expressed using the same base. A more general method utilizes our calculators and the third logarithm law:

THE THIRD LOGARITHM LAW

$$\log_b\left(a^x\right) = x\log_b a$$

***Exercise* #1:** Solve: $4^x = 8$ using (a) common bases and (b) the logarithm law shown above.

(a) Method of Common Bases

(b) Logarithm Approach

The beauty of this logarithm law is that it removes the variable from the exponent. This law, in combination with the logarithm base 10, the **common log**, allows us to solve almost any exponential equation using calculator technology.

***Exercise* #2:** Solve each of the following equations for the value of *x*. Round your answers to the nearest *hundredth*.

(a) $5^x = 18$

(b) $4^x = 100$

(c) $2^x = 1560$

These equations can become more complicated, but each and every time we will use the logarithm law to transform an exponential equation into one that is more familiar (linear only for now)

***Exercise* #3:** Solve each of the following equations for *x*. Round your answers to the nearest *hundredth*.

(a) $6^{x+3} = 50$

(b) $(1.03)^{\frac{x}{2}-5} = 2$

Now that we are familiar with this method, we can revisit some of our exponential models from earlier in the unit. Recall that for an exponential function that is growing:

> If quantity Q is known to increase by a fixed percentage p, in decimal form, then Q can be modeled by
>
> $$Q(t)=Q_0(1+p)^t$$
>
> where Q_0 represents the amount of Q present at $t=0$ and t represents time.

***Exercise* #4:** A biologist is modeling the population of bats on a tropical island. When he first starts observing them, there are 104 bats. The biologist believes that the bat population is growing at a rate of 3% per year.

(a) Write an equation for the number of bats, $B(t)$, as a function of the number of years, t, since the biologist started observing them.

(b) Using your equation from part (a), algebraically determine the number of years it will take for the bat population to reach 200. Round your answer to the nearest year.

***Exercise* #5:** A stock has been declining in price at a steady pace of 5% per week. If the stock started at a price of $22.50 per share, determine algebraically the number of weeks it will take for the price to reach $10.00. Round your answer to the nearest week.

As a final discussion, we return to evaluating logarithms using our calculator. Many modern calculators can find a logarithm of any base. Some still only have the common log (base 10) and another that we will soon see. But, we can still express our answers in terms of logarithms.

***Exercise* #6:** Find the solution to each of the following exponential equations in terms of a logarithm with the same base as the exponential equation.

(a) $4(2)^x-3=17$

(b) $17(5)^{x/3}=4$

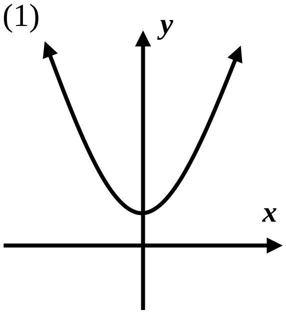
5- 9 The Number e and the Natural Logarithm
Math III

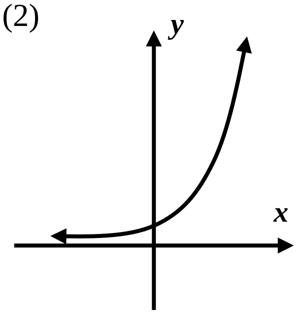
There are many numbers in mathematics that are more important than others because they find so many uses in either mathematics or science. Good examples of important numbers are 0, 1, i, and π. In this lesson you will be introduced to an important number given the letter **e** for its "inventor" Leonhard Euler (1707-1783). This number plays a crucial role in Calculus and more generally in modeling exponential phenomena.

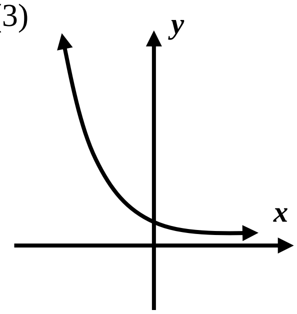
The Number e

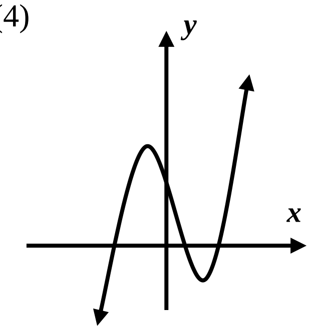
1. Like π, e is irrational. 2. $e \approx 2.72$ 3. Used in Exponential Modeling

***Exercise* #1:** Which of the graphs below shows $y = e^x$? Explain your choice. Check on your calculator.

(1)


(2)


(3)


(4)


Explanation:

____

Very often e is involved in exponential modeling of both increasing and decreasing quantities. The creation of these models is beyond the scope of this course, but we can still work with them.

***Exercise* #2:** A population of llamas on a tropical island can be modeled by the equation $P = 500e^{0.035t}$, where t represents the number of years since the llamas were first introduced to the island.

(a) How many llamas were initially introduced at $t = 0$? Show the calculation that leads to your answer.

(b) Algebraically determine the number of years for the population to reach 600. Round your answer to the nearest *tenth* of a year.

Because of the importance of $y = e^x$, its **inverse**, known as the **natural logarithm**, is also important.

THE NATURAL LOGARITHM

The inverse of $y = e^x$: $y = \ln x \quad (y = \log_e x)$

The natural logarithm, like all logarithms, gives an exponent as its output. In fact, it gives the power that we must raise **e** to in order to get the input.

***Exercise* #3:** *Without* the use of your calculator, determine the values of each of the following.

(a) $\ln(e)$ (b) $\ln(1)$ (c) $\ln(e^5)$ (d) $\ln\sqrt{e}$

The natural logarithm follows the three basic logarithm laws that all logarithms follow. The following problems give additional practice with these laws.

***Exercise* #4:** Which of the following is equivalent to $\ln\left(\frac{x^3}{e^2}\right)$?

(1) $\ln x + 6$ (3) $3\ln x - 6$

(2) $3\ln x - 2$ (4) $\ln x - 9$

____

***Exercise* #5:** A hot liquid is cooling in a room whose temperature is constant. Its temperature can be modeled using the exponential function shown below. The temperature, *T*, is in degrees Fahrenheit and is a function of the number of minutes, *m*, it has been cooling.

$$T(m) = 101e^{-0.03m} + 67$$

(a) What was the initial temperature of the water at $m = 0$. Do without using your calculator.

(b) How do you interpret the statement that $T(60) = 83.7$?

(c) Using the natural logarithm, determine algebraically when the temperature of the liquid will reach $100\ ^{o}\text{F}$. Show the steps in your solution. Round to the nearest tenth of a minute.

(d) On average, how many degrees are lost per minute over the interval $10 \le m \le 30$? Round to the nearest tenth of a degree.

5-10 COMPOUND INTEREST
MATH III

In the worlds of investment and debt, interest is added onto a principal in what is known as **compound interest**. The percent rate is typically given on a yearly basis, but could be applied more than once a year. This is known as the **compounding frequency**. Let's take a look at a typical problem to understand how the compounding frequency changes how interest is applied.

***Exercise* #1:** A person invests $500 in an account that earns a **nominal yearly interest rate** of 4%.

(a) How much would this investment be worth in 10 years if the **compounding frequency** was once per year? Show the calculation you use.

(b) If, on the other hand, the interest was applied four times per year (known as quarterly compounding), why would it not make sense to multiply by 1.04 each quarter?

(c) If you were told that an investment earned 4% per year, how much would you assume was earned per quarter? Why?

(d) Using your answer from part (c), calculate how much the investment would be worth after 10 years of quarterly compounding? Show your calculation.

So, the pattern is fairly straightforward. For a **shorter compounding period**, we get to **apply the interest more often**, but at a **lower rate**.

***Exercise* #2:** How much would $1000 invested at a nominal 2% yearly rate, compounded monthly, be worth in 20 years? Show the calculations that lead to your answer.

(1) $1485.95 (3) $1033.87

(2) $1491.33 (4) $1045.32 __________

This pattern is formalized in a classic formula from economics that we will look at in the next exercise.

***Exercise* #3:** For an investment with the following parameters, write a formula for the amount the investment is worth, *A*, after *t*-years.

P = amount initially invested

r = nominal yearly rate

n = number of compounds per year

$A(t) =$

The rate in *Exercise* #1 was referred to as **nominal (in name only)**. It's known as this, because you effectively earn more than this rate if the compounding period is more than once per year. Because of this, bankers refer to the **effective rate**, or the rate you would receive if compounded just once per year. Let's investigate this.

***Exercise* #4:** An investment with a nominal rate of 5% is compounded at different frequencies. Give the **effective** yearly rate, accurate to two decimal places, for each of the following compounding frequencies. Show your calculation.

(a) Quarterly

(b) Monthly

(c) Daily

We could compound at smaller and smaller frequency intervals, eventually compounding all moments of time. In our formula from *Exercise* #3, we would be letting n approach infinity. Interestingly enough, this gives rise to **continuous compounding** and the use of the natural base **e** in the famous **continuous compound interest formula.**

CONTINUOUS COMPOUND INTEREST

For an initial principal, P, compounded continuously at a nominal yearly rate of r, the investment would be worth an amount A given by:

$$A(t) = Pe^{rt}$$

***Exercise* #5:** A person invests $350 in a bank account that promises a nominal rate of 2% continuously compounded.

(a) Write an equation for the amount this investment would be worth after t-years.

(b) How much would the investment be worth after 20 years?

(c) Algebraically determine the time it will take for the investment to reach $400. Round to the nearest tenth of a year.

(d) What is the effective annual rate for this investment? Round to the nearest hundredth of a percent.