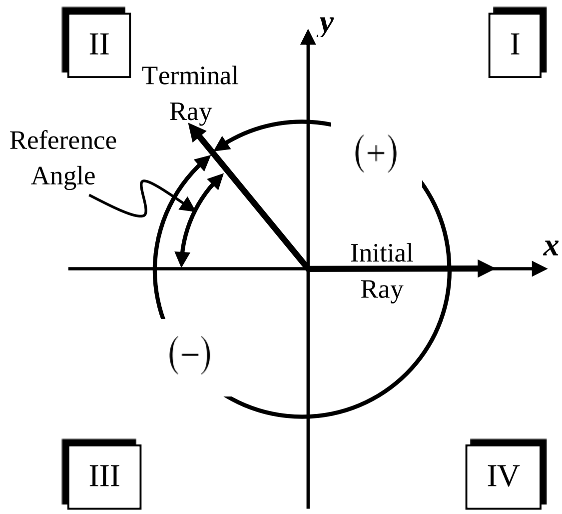
7- 1 ROTATIONS AND ANGLE TERMINOLOGY

In this unit we will be studying the three basic **trigonometric functions**. These functions are based on the geometry of a circle and rotations around its center. Sometimes the trigonometric functions are known as **circular functions.** In this introductory lesson we introduce some basic terminology and concepts concerning angles. Some of the terminology is specified below.



Standard Position: An angle is said to be drawn in standard position if its vertex is at the origin and its initial ray points along the positive *x*-axis.

Positive and Negative Rotations: A rotation is said to be positive if the initial ray is rotated counter-clockwise to the terminal ray and said to be negative if the initial ray is rotated clockwise to the terminal ray.

Coterminal Anlges: Any two angles drawn in standard position that share a terminal ray.

Reference Angles: The positive acute angle formed by the terminal ray and the *x*-axis.

***Exercise* #1:** For each of the following angles, given by the Greek letter **theta**, draw a rotation diagram and identify the quadrant that the terminal ray falls in.

(a) $\theta = 145^\circ$ (b) $\theta = 320^\circ$ (c) $\theta = 72^\circ$ (d) $\theta = -210^\circ$

(e) $\theta = 250^\circ$ (f) $\theta = -310^\circ$ (g) $\theta = 460^\circ$ (h) $\theta = -400^\circ$

***Exercise* #2:** In which quadrant would the terminal ray of an angle drawn in standard position fall if the angle measures 860°?

(1) I (3) III

(2) II (4) IV

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***Exercise* #3:** Give a negative angle that is coterminal with each of the following positive angles, **alpha**.

(a) $\alpha = 90^\circ$ (b) $\alpha = 330^\circ$ (c) $\alpha = 120^\circ$ (d) $\alpha = 210^\circ$

***Exercise* #4:** Coterminal angles drawn in standard position will always have measures that differ by an integer multiple of

(1) 90° (3) 180°

(2) 360° (4) 720°

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***Exercise* #5:** For each of the following angles, beta, draw a rotation diagram and then state **beta's** reference angle, β_r.

(a) $\beta = 160^\circ$ (b) $\beta = 300^\circ$ (c) $\beta = 210^\circ$ (d) $\beta = 78^\circ$

(e) $\beta = -110^\circ$ (f) $\beta = -280^\circ$ (g) $\beta = 605^\circ$ (h) $\beta = -410^\circ$

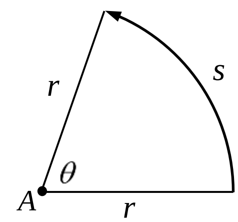
7-2 RADIAN ANGLE MEASUREMENT

Just as distance can be measured in inches, feet, miles, centimeters, etcetera, rotations about a point can also be measured in many different ways. Measuring one complete rotation in terms of 360° is somewhat arbitrary. A common unit of angle measurement that is an alternative to degrees is called the **radian**. You first saw this unit of angle in Common Core Geometry. It is natural to define it in terms of the geometry of a circle, given the unit we are in:

THE DEFINITION OF A RADIAN

The radian angle, θ, created by a rotation about a point *A* using a radius of *r* and passing through an arc length of *s* is defined as

$$\theta = \frac{s}{r} \quad \text{or equivalently} \quad s = \theta \cdot r$$



***Exercise* #1:** Consider a full rotation around any point in the counter-clockwise (positive) direction.

(a) What is the arc length, *s*, in terms of the radius of the circle, *r*, for a full rotation?

(b) Based on the definition above and on your answer to part (a), how many radians are there in one full rotation?

Radians essentially measure the total number of radii (hence the name) that have been traversed about the circumference of a circle in a given rotation. Based on the circumference formula of a circle, thus, there will always be 2π radians in one full rotation.

***Exercise* #2:** Use the formula above to answer each of the following.

(a) Determine the number of radians that the minute hand of a clock passes through if it has a length of 5 inches and its tip travels a total distance of 13 inches.

(b) If a pendulum swings through an angle of 0.55 radians, what distance does its tip travel if it has a length of 8 feet?

Radians are essential in the study of higher-level mathematics and physics and are the angle measurement of choice for the study of calculus. It is important to be able to convert between the angular system of degrees and that of radians. The next exercise will illustrate this process.

***Exercise* #3:** Consider one-half of a full rotation.

(a) What is the angle of rotation in both degrees and in radians?

(b) Using the two equivalent angles of rotation in (a), convert a 30° angle into an equivalent angle in radians.

***Exercise* #4:** Convert each of the following common angles in degrees into radians. Express your answers in terms of pi.

(a) $\theta = 90^{\circ}$ (b) $\theta = 120^{\circ}$ (c) $\theta = 225^{\circ}$

***Exercise* #5:** Convert each of the following common radian angles into degrees.

(a) $\theta = \frac{5\pi}{6}$ (b) $\theta = \frac{3\pi}{2}$ (c) $\theta = \frac{3\pi}{4}$

We should also feel comfortable with the fact that radians do not always have to be in terms of pi, although they often are.

***Exercise* #6:** Convert each of the following radian angles, which aren't in terms of pi, into degrees. Round your answers to the nearest degree.

(a) $\theta = 5.8$ (b) $\theta = 4.2$ (c) $\theta = -2.5$

***Exercise* #7:** An angle drawn in standard position whose radian measure is 2 radians would terminate in which of the following quadrants?

(1) I (3) III

(2) II (4) IV

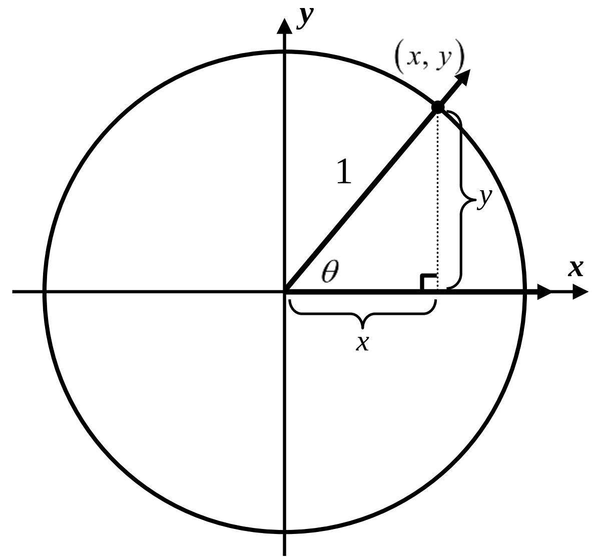
7-3 THE DEFINITION OF THE SINE AND COSINE FUNCTIONS

The sine and cosine functions form the basis of trigonometry. We would like to define them so that their definition is consistent with what you already are familiar with concerning right triangle trigonometry. Recall from Common Core Geometry that in a right triangle the sine and cosine ratios were defined as:

$$\sin A = \frac{\text{side length opposite of } A}{\text{length of the hypotenuse}} \quad \text{and} \quad \cos A = \frac{\text{side length adjacent to } A}{\text{length of the hypotenuse}}$$

***Exercise* #1:** Consider the **unit circle** shown below with an angle, θ, drawn in standard position.

(a) Given the right triangle shown, find an expression for $\sin(\theta)$.



(b) Given the right triangle shown, find an expression for $\cos(\theta)$.

We thus define the sine and cosine functions by using the coordinates on the unit circle. They are the first functions that are **geometrically defined** as they are based on the geometry of a circle (circular functions).

THE DEFINITION OF THE SINE AND COSINE FUNCTIONS

For an angle in standard position whose terminal ray passes through the point (x, y) on the unit circle:

$$\sin(\theta) = \text{ the } y\text{-coordinate} \quad \text{and} \quad \cos(\theta) = \text{the } x\text{-coordinate}$$

The above definition is **unquestionably the most important fact to memorize** concerning trigonometry. We can now use this along with our work on the unit circle to determine certain **exact** values of cosine and sine.

***Exercise* #2:** Using the unit circle diagram, determine each of the following values.

(a) $\sin(30^\circ) =$ (b) $\sin(240^\circ) =$ (c) $\cos(90^\circ) =$ (d) $\cos(180^\circ) =$

(e) $\sin(90^\circ) =$ (f) $\sin(135^\circ) =$ (g) $\cos(150^\circ) =$ (h) $\cos(0^\circ) =$

***Exercise* #3:** The terminal ray of an angle, α, drawn in standard position passes through the point $(-0.6, 0.8)$, which lies on the unit circle. Which of the following gives the value of $\sin(\alpha)$?

(1) 1.2 (3) -0.6

(2) 0.8 (4) 0.2 ________

It is important to be able to determine the sign (positive or negative) of each of the two basic trigonometric functions for an angle whose terminal ray lies in a given quadrant. The next exercise illustrates this process.

***Exercise* #4:** For each quadrant below, determine if the sine and cosine of an angle whose terminal ray falls in the quadrant is positive $(+)$ or negative $(-)$.

	I	II	III	IV
$\sin(\theta)$				
$\cos(\theta)$				

Since each point on the unit circle must satisfy the equation $x^2 + y^2 = 1$, we can now state what is known as the **Pythagorean Identity**.

THE PYTHAGOREAN IDENTITY

For any angle, θ, $(\cos\theta)^2 + (\sin\theta)^2 = 1$

***Exercise* #5:** An angle, α, has a terminal ray that falls in the second quadrant. If it is known that $\sin(\alpha) = \frac{3}{5}$, determine the value of $\cos(\alpha)$.

***Exercise* #6:** An angle, θ, has a terminal ray that falls in the first quadrant and $\cos(\theta) = \frac{1}{3}$. Determine the value of $\sin(\theta)$ in simplest radical form.

More Work with the Sine and Cosine Functions

The sine and cosine functions are the first a student typically encounters that are **non-algebraic**, that is they cannot be thought of as combinations of a finite number of integer powers and/or roots. Since they are defined by using the geometry of a circle they are not all that intuitive. Additional practice will be given in this lesson to simply get used to them.

***Exercise* #1:** Recall the following definitions of the sine and cosine functions. If θ is an angle drawn in standard position whose terminal ray passes through the point (x, y) on the unit circle then …

$$\sin(\theta) = \qquad \text{and} \qquad \cos(\theta) =$$

***Exercise* #2:** Given the function $f(x) = 6\sin(x)$ which of the following is the value of $f(60^\circ)$?

(1) $4\sqrt{2}$ (3) 3

(2) $3\sqrt{3}$ (4) 0

***Exercise* #3:** If $g(\alpha) = 4\cos(\alpha) - 2\sin(\alpha)$ then $g(330^\circ) = ?$

(1) $8\sqrt{2}+3$ (3) $2\sqrt{3}+1$

(2) $4\sqrt{3}-2$ (4) $6\sqrt{2}-4$

***Exercise* #4:** Which of the following is not equal to one?

(1) $\cos(0^\circ)$ (3) $\sin(90^\circ)$

(2) $\cos(360^\circ)$ (4) $\sin(270^\circ)$

***Exercise* #5:** For an angle α whose terminal ray lies in the third quadrant it is known that $\cos(\alpha) = -0.96$. Which of the following is the value of $\sin(\alpha)$?

(1) -0.28 (3) 0.04

(2) -0.56 (4) 0.78

A special relationship exists between the trigonometric values of an angle and those of its reference angle. This important relationship is illustrated in the next exercise.

***Exercise* #6:** In each of the following, an angle and its reference have been given. Using your calculator, in **degree mode**, determine the sine and cosine of **both** the angle and its reference. Round all answers to the nearest *hundredth*.

(a) $\theta = 110^\circ$ and $\theta_r = 70^\circ$ (b) $\theta = 235^\circ$ and $\theta_r = 55^\circ$ (c) $\theta = 282^\circ$ and $\theta_r = 78^\circ$

Clearly the absolute value of the sine and cosine are the same for an angle and its reference. This fact can be exploited to produce sine and cosine values for angles if they are known for their references.

***Exercise* #7:** Given that $\cos\left(30^\circ\right) = \sqrt{3}/2$ and $\sin\left(30^\circ\right) = 1/2$, determine the following values in exact form.

(a) $\cos\left(150^\circ\right)$ (b) $\sin\left(150^\circ\right)$ (c) $\cos\left(210^\circ\right)$

(d) $\sin\left(210^\circ\right)$ (e) $\cos\left(330^\circ\right)$ (f) $\sin\left(330^\circ\right)$

We should not forget that the trigonometric functions are valid for radians as well as degrees. Practice evaluating these functions for each of the following radian inputs. If needed, convert to degrees.

***Exercise* #8:** Evaluate each of the following trigonometric expressions.

(a) $\sin\left(\frac{\pi}{2}\right)$ (b) $\sin\left(\frac{\pi}{3}\right)$ (c) $\cos\left(\frac{3\pi}{2}\right)$ (d) $\cos\left(\frac{3\pi}{4}\right)$

7-4 BASIC GRAPHS OF SINE AND COSINE
COMMON CORE ALGEBRA II

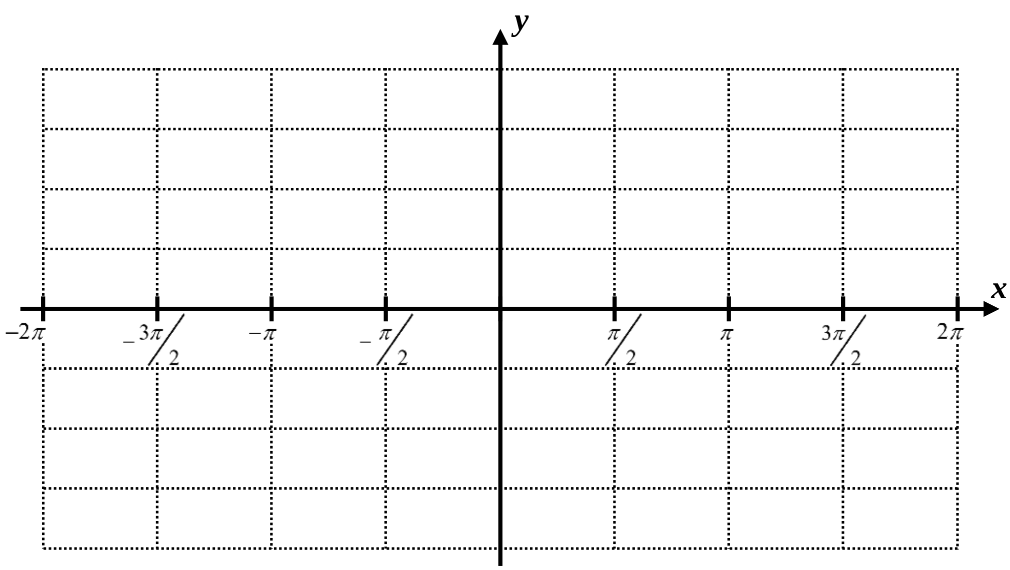
The sine and cosine functions can be easily graphed by considering their values at the quadrantal angles, those that are integer multiples of 90° or $\frac{\pi}{2}$ radians. Due to considerations from physics and calculus, most trigonometric graphing is done with the input angle in units of radians, not degrees.

***Exercise* #1:** Consider the functions $f(x)=\sin(x)$ and $g(x)=\cos(x)$, where x is an angle in radians.

(a) By using the unit circle, fill out the table below for selected quadrantal angles.

x	-2π	$-3\pi/2$	$-\pi$	$-\pi/2$	0	$\pi/2$	π	$3\pi/2$	2π
$\cos(x)$									
$\sin(x)$									

(b) Graph both the sine and cosine curves on the grid shown below. **Clearly** label which curve is which.



(c) The domain and range of the sine and cosine functions are the same. State them below in interval notation.

Domain: Range:

(d) After how much horizontal distance will both sine and cosine repeat its basic pattern? This is called the **period** of the trigonometric graph. Because these graphs have patterns that repeat they are called **periodic**.

Now we would like to explore the effect of changing the coefficient of the trigonometric function. In essence we would like to look at the graphs of functions of the forms:

$$y = A\sin(x) \quad \text{and} \quad y = A\cos(x)$$

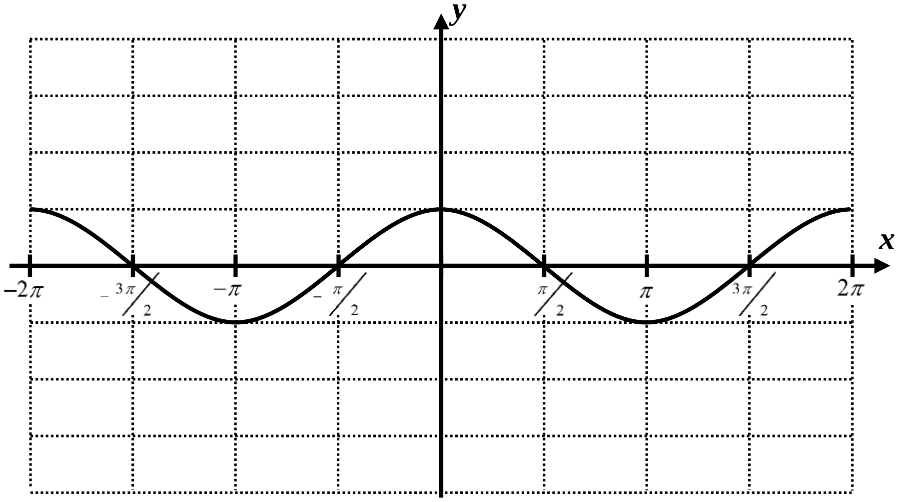
***Exercise* #2:** The grid below shows the graph of $y = \cos(x)$. Use your graphing calculator to sketch and label each of the following equations. Be sure your calculator is in **RADIAN MODE**.

$y = 3\cos(x)$

$y = -4\cos(x)$

$y = \frac{3}{2}\cos(x)$

$y = -2\cos(x)$



As we can see, this coefficient controls the height that the cosine curves rises and falls above the *x*-axis. Its absolute value is given the name **amplitude**. In terms of sound waves it indicates the volume of the sound.

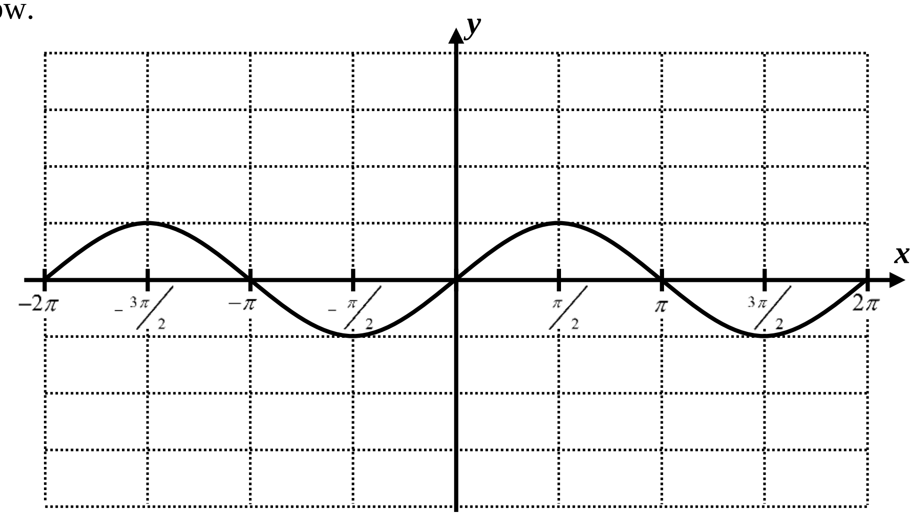
***Exercise* #4:** The basic sine function is graphed below. **Without** the use of your calculator, sketch each of the following sine curves on the axes below.

$y = 2\sin(x)$

$y = 4\sin(x)$

$y = -3\sin(x)$

$y = -\frac{1}{2}\sin(x)$



7-5 VERTICAL SHIFTING OF SINUSOIDAL GRAPHS

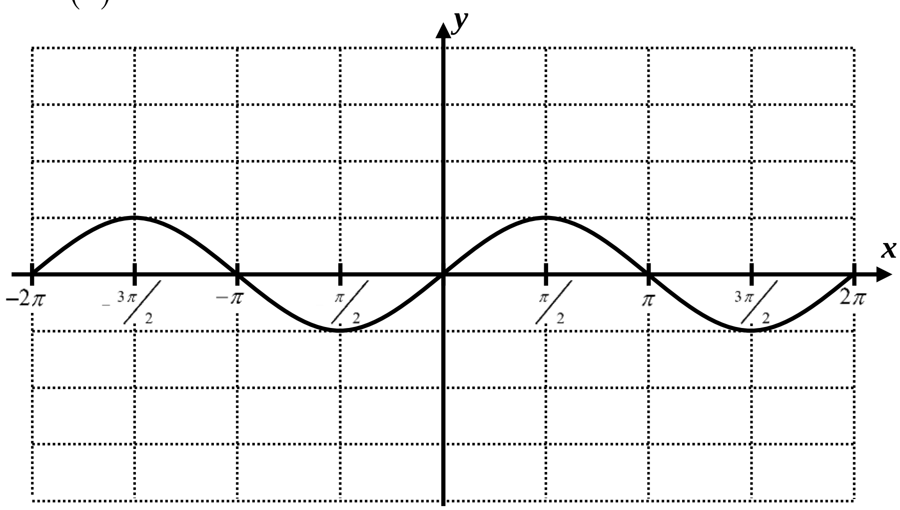
Any graph primarily comprised of either the sine or cosine function is known as **sinusoidal**. These graphs can be stretched vertically, as we saw in the last lesson. Other transformations can occur as well. Today we will explore graphs of equations of the form:

$$y = A\sin(x) + C \quad \text{and} \quad y = A\cos(x) + C$$

Since we already understand the effect of *A* on the graph, it is now time to review the effect of adding a constant to an equation.

***Exercise* #1:** Consider the function $f(x) = \sin(x) + 3$.

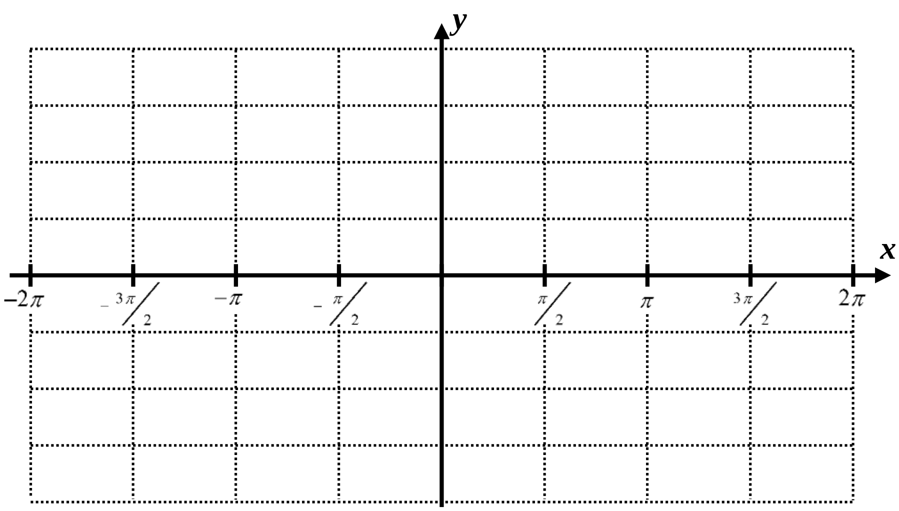
(a) How would the graph of $y = \sin(x)$ be shifted to produce the graph of $f(x)$?



(b) On the grid to the right is the basic sine curve, $y = \sin(x)$. On the same grid, sketch the graph of $f(x)$.

***Exercise* #2:** Consider the function $y = 2\cos(x) + 1$.

(a) Using your calculator, sketch the graph on the grid to the right.



(b) Give the equation of a horizontal line that this curve rises and falls two units above. Sketch this line on the graph.

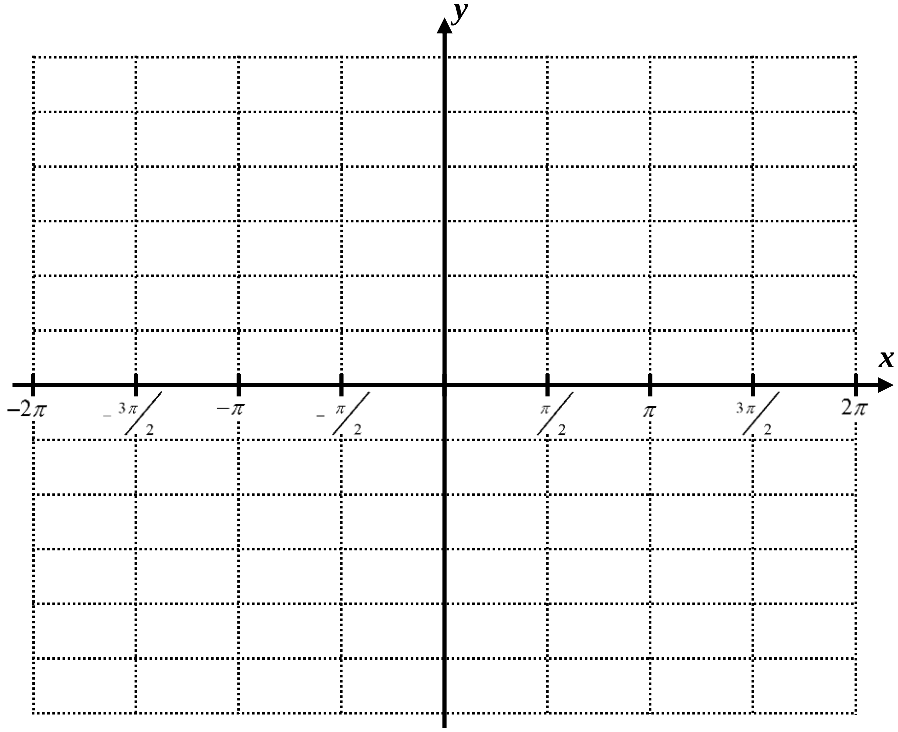
(c) State the range of this trigonometric function in interval notation.

For curves that have the general form $y = A\sin(x) + C$ and $y = A\cos(x) + C$ the value C is called the **midline** or **average value** of the trigonometric function. It is the height or horizontal line that the sinusoidal curve rises and falls above and below by a distance of $|A|$ (the amplitude).

***Exercise* #3:** Sketch and label the functions $y = 4\sin(x) - 2$ and $y = -2\cos(x) + 3$ on the grid below. Try them first without your calculator and then use it to help or verify your graphs. Then, state the ranges of each of the equations in interval notation.

Range of $y = 4\sin(x) - 2$:

Range of $y = -2\cos(x) + 3$:



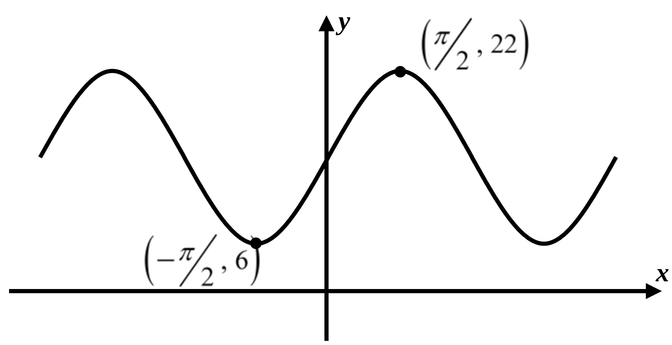
***Exercise* #4:** Determine the range of each of the following trigonometric functions. Express your answer in interval notation.

(a) $y = 7\sin(x) + 4$

(b) $y = -5\cos(x) + 2$

(c) $y = 25\sin(x) + 35$

***Exercise* #5:** The graph below shows a sinusoidal curve of the form $y = A\sin(x) + C$. Determine the values of A and C. Show how you arrived at your results.

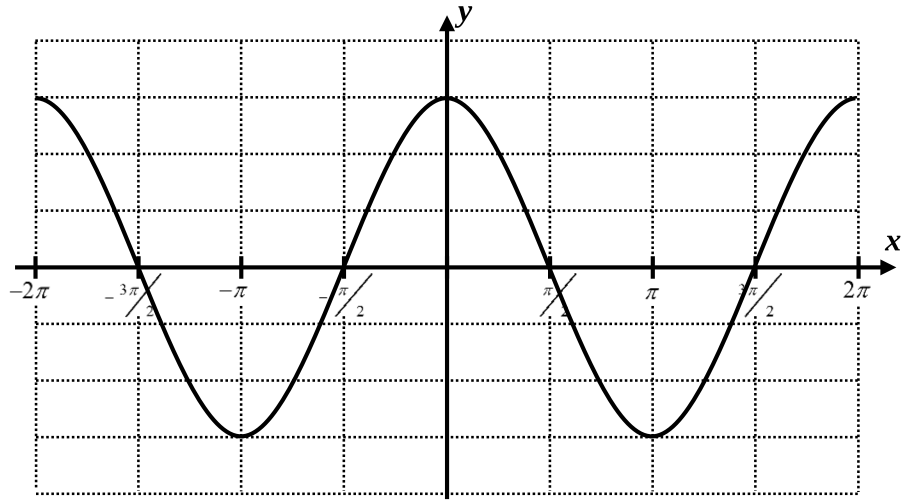


7-6 THE FREQUENCY AND PERIOD OF A SINUSOIDAL GRAPH

A final transformation will allow us to horizontally stretch and compress sinusoidal graphs. It is important to be able to do this, especially when modeling real-world phenomena, because most **periodic functions** do not have a period of 2π. The first exercise will illustrate the pattern.

***Exercise* #1:** On the grid below is a graph of the function $y = 3\cos(x)$.

(a) Using your calculator, sketch the graph of $y = 3\cos(2x)$ on the same axes.



(b) How many full cycles or periods of this function now fit within 2π radians?

(c) Using your calculator, sketch the graph of $y = 3\cos\left(\frac{1}{2}x\right)$ on the same axes.

(d) How many full cycles or periods of this function now fit within 2π radians?

The **period**, P, of a sinusoidal function is an extremely important concept. It is defined as **the minimum horizontal shift needed for the function to repeat its fundamental pattern**. The period for the basic sinusoidal graphs is 2π. Clearly, from our first exercise, the period of the function depends on the coefficient B in the general equations $y = A\sin(Bx)$ and $y = A\cos(Bx)$. This coefficient, B, is known as the **frequency**.

***Exercise* #2:** Consider the graphs from *Exercise* #1. For each below, state the frequency and period.

(a) $y = 3\cos(x)$

Frequency, $B =$

Period, $P =$

(b) $y = 3\cos(2x)$

Frequency, $B =$

Period, $P =$

(c) $y = 3\cos\left(\frac{1}{2}x\right)$

Frequency, $B =$

Period, $P =$

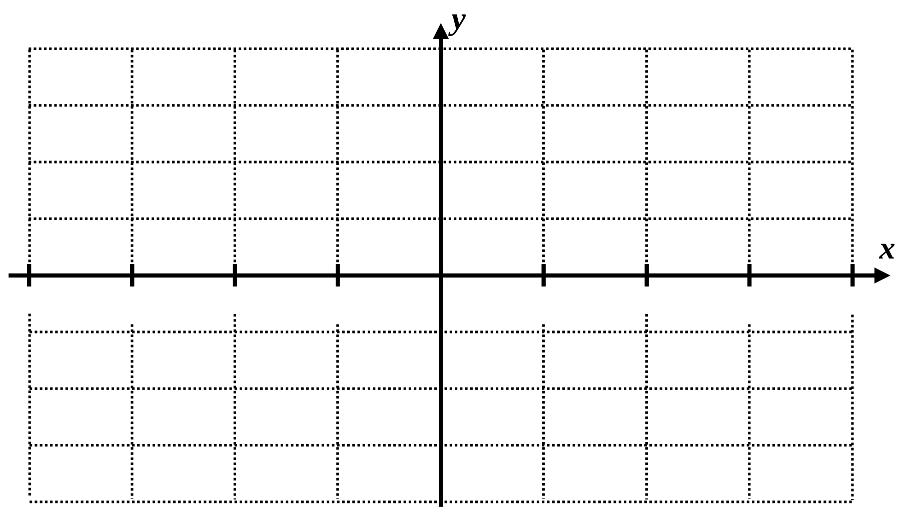
Clearly we can see from *Exercise* #2 that the frequency and period are **inversely related**, that is as one increases the other decreases and vice versa.

***Exercise* #3:** Examine the results from *Exercise* #2. What is true about the product of the period, P, and the frequency, B? Write an equation for this relationship.

***Exercise* #4:** Determine the period of each of the following sinusoidal functions. Express your answers in exact form.

(a) $y = 6\sin(4x)$ (b) $y = 8\cos\left(\frac{\pi}{3}x\right)$ (c) $y = -12\sin\left(\frac{2}{3}x\right)$

***Exercise* #5:** Sketch the function $y = 2\sin(4x)$ on the grid below for one full period to the left and right of the y-axis. Label the scale on your axes.



***Exercise* #6:** The heights of the tides can be described using a sinusoidal model of the form $y = A\cos(Bx) + C$. If high tides are separated by 24 hours, which of the following gives the frequency, B, of the curve?

(1) 12 (3) $\frac{\pi}{12}$

(2) $\frac{\pi}{24}$ (4) $\frac{\pi}{6}$

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7-7 SINUSOIDAL MODELING

The sine and cosine functions can be used to model a variety of real-world phenomena that are periodic, that is, they repeat in predictable patterns. The key to constructing or interpreting a sinusoidal model is understanding the physical meanings of the coefficients we've explored in the last three lessons.

SINUSOIDAL MODEL COEFFICIENTS

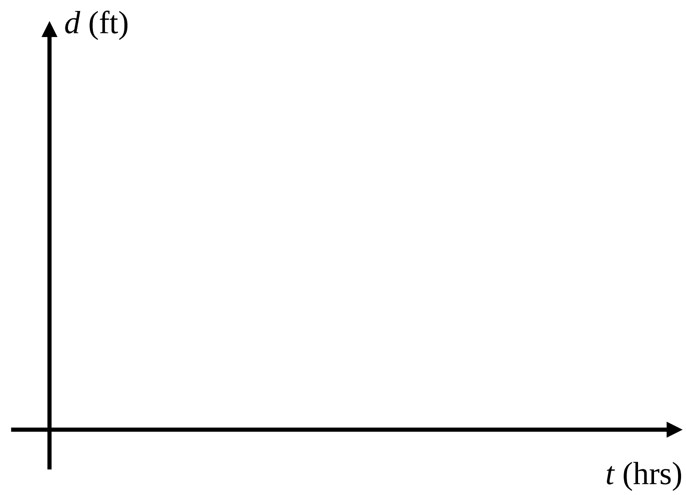
For $y = A\sin\left(Bx\right) + C$ and $y = A\cos\left(Bx\right) + C$

$\lvert A \rvert$	the **amplitude** or distance the sinusoidal model rises and falls above its midline
C	the **midline** or average *y*-value of the sinusoidal model
B	the **frequency** of the sinusoidal model – related to the **period**, *P*, by the equation $BP = 2\pi$
P	the **period** of the sinusoidal model – the minimum distance along the *x*-axis for the cycle to repeat

***Exercise* #1:** The tides in a particular bay can be modeled with an equation of the form $d = A\cos\left(Bt\right) + C$, where t represents the number of hours since high-tide and d represents the depth of water in the bay. The maximum depth of water is 36 feet, the minimum depth is 22 feet and high-tide is hit every 12 hours.

(a) On the axes, sketch a graph of this scenario for two full periods. Label the points on this curve that represent high and low tide.

(b) Determine the values of *A*, *B*, and *C* in the model. Verify your answers and sketch are correct on your calculator.



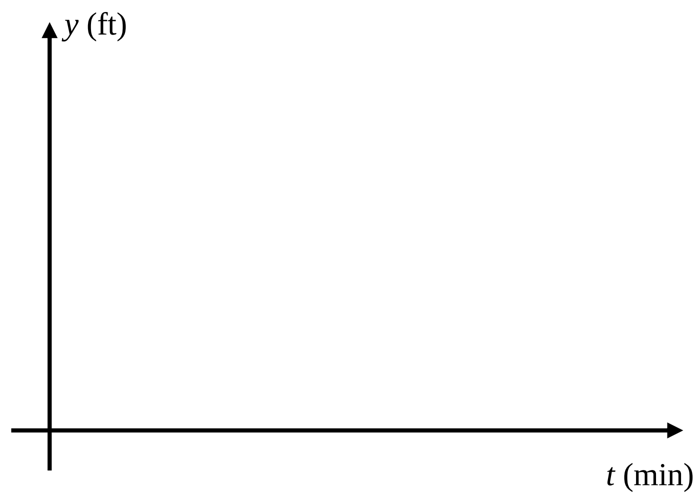
(c) Tanker boats cannot be in the bay when the depth of water is less or equal to 25 feet. Set up an inequality and solve it graphically to determine all points in time, t, on the interval $0 \le t \le 24$ when tankers cannot be in the bay. Round all times to the nearest *tenth* of an hour.

***Exercise* #2:** The height of a yo-yo above the ground can be well modeled using the equation $h = 1.75\cos(\pi t) + 2.25$, where *h* represents the height of the yo-yo in feet above the ground and *t* represents time in seconds since the yo-yo was first dropped from its maximum height.

(a) Determine the maximum and minimum heights that the yo-yo reaches above the ground. Show the calculations that lead to your answers.

(b) How much time does it take for the yo-yo to return to the maximum height for the first time?

***Exercise* #3:** A Ferris wheel is constructed such that a person gets on the wheel at its lowest point, five feet above the ground, and reaches its highest point at 130 feet above the ground. The amount of time it takes to complete one full rotation is equal to 8 minutes. A person's vertical position, *y*, can be modeled as a function of time in minutes since they boarded, *t*, by the equation $y = A\cos(Bt) + C$. Sketch a graph of a person's vertical position for one cycle and then determine the values of *A*, *B*, and *C*. Show the work needed to arrive at your answers.



***Exercise* #4:** The possible hours of daylight in a given day is a function of the day of the year. In Poughkeepsie, New York, the minimum hours of daylight (occurring on the Winter solstice) is equal to 9 hours and the maximum hours of daylight (occurring on the Summer solstice) is equal to 15 hours. If the hours of daylight can be modeled using a sinusoidal equation, what is the equation's ampltitude?

(1) 6 (3) 3

(2) 12 (4) 4